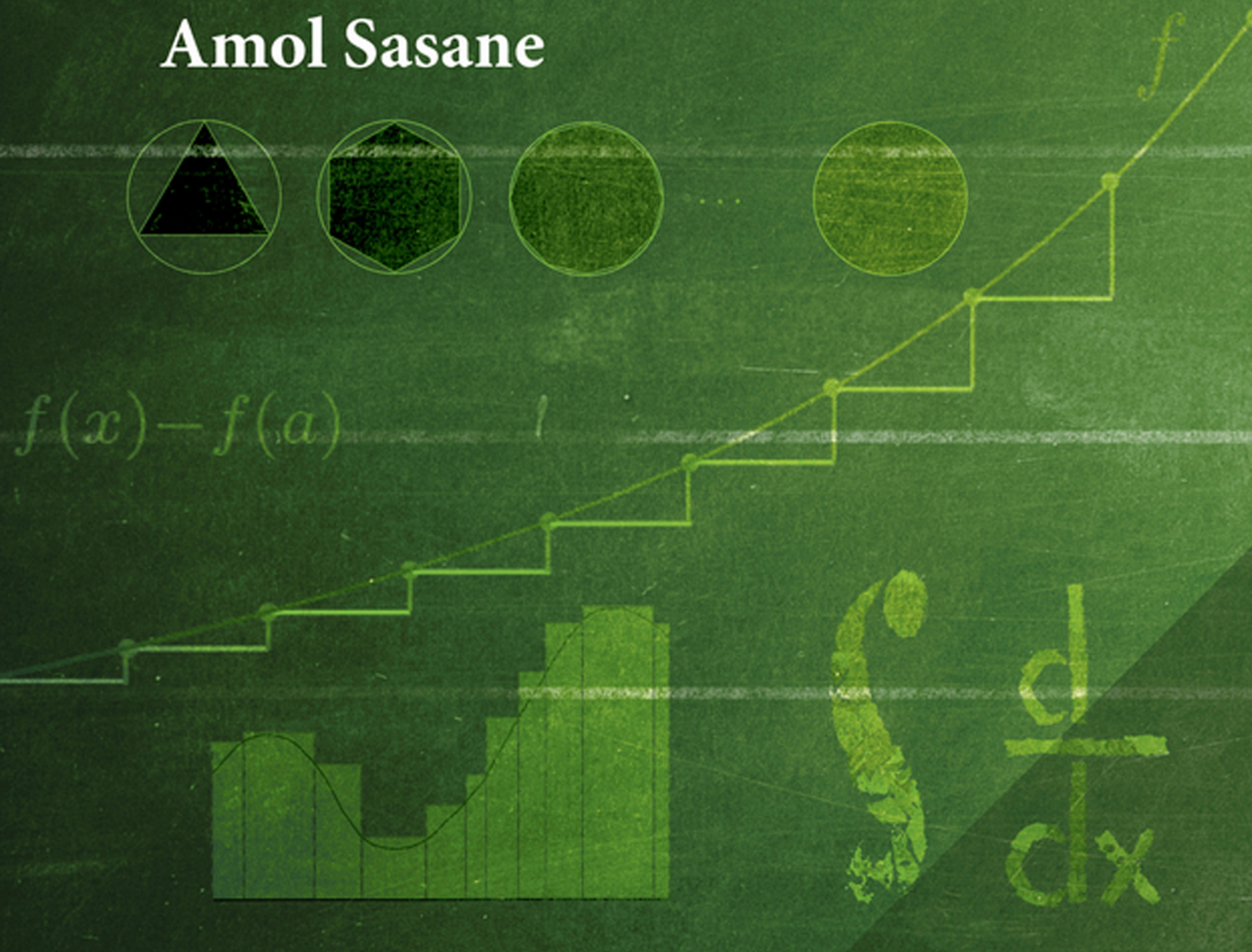
The How and Why of One Variable CALCULUS

Amol Sasane



WILEY

The How and Why of
One Variable Calculus

The How and Why of One Variable Calculus

Amol Sasane

London School of Economics, UK

WILEY

This edition first published 2015

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com.

Library of Congress Cataloging-in-Publication Data applied for.

ISBN: 9781119043386

A catalogue record for this book is available from the British Library.

Cover Image: Tuomas Kujansuu/iStockphoto

Set in 10/12pt, TimesLtStd by SPi Global, Chennai, India

1 2015

To my parents

Contents

Preface

Who is this book is for?

This book is meant as a textbook for an honours course in Calculus, and is aimed at first year students beginning studies at the university. The preparation assumed is high school level Mathematics. Any arguments not met before in high school (for example, geometric arguments à la Euclid) can be picked up along the way or simply skipped without any loss of continuity. This book may also be used as supplementary reading in a traditional methods-based Calculus course or as a textbook for a course meant to bridge the gap between Calculus and Real Analysis.

How should the student read the book?

Students reading the book should not feel obliged to study every proof at the first reading. It is more important to understand the theorems well, to see how they are used, and why they are interesting, than to spend all the time on proofs. So, while reading the book, one may wish, after reading the theorem statement, to first study the examples and solve a few relevant exercises, before returning to read the proof of that theorem.

The exercises are an integral part of studying this book. They are a combination of purely drill ones (meant for practising Calculus methods), and those meant to clarify the meanings of the definitions, theorems, and even to facilitate the goal of developing 'mathematical maturity'. The student should feel free to skip exercises that seem particularly challenging at the first instance, and return back to them now and again. Although detailed solutions are provided, the student should not be tempted to consult the given solution too soon. In the learning process leading to developing understanding, it is much better to think about the exercise (even if one does not find the answer oneself!), rather than look at the provided solution in order to understand how to solve it. In other words, it is the *struggle* to solve the exercise that turns out to be more important than the mere *knowledge* of the solution. After all, given a *new* problem, it will be the struggle that pays off, and not the knowledge of the solution of the (now irrelevant) *old* exercise! So the student should absolutely not feel discouraged if he or she doesn't manage to solve an exercise problem. Some of the exercises that are more abstract/technical/challenging as compared to the other exercises are indicated with an asterisk symbol ($*$).

Acknowledgements

I would like to thank Sara Maad Sasane (Lund University) for going through the entire manuscript, pointing out typos and mistakes, and offering insightful suggestions and comments. Thanks are also due to Lassi Paunonen (Tampere University) and Raymond Mortini (University of Lorraine-Metz) for many useful comments. A few pedagogical ideas in this book stem from some of the references listed at the end of this book. This applies also to the exercises. References are given in the section on notes at the end of the chapters, but no claim to originality is made in case there is a missing reference. The figures in this book have been created using xfig, Maple, and MATLAB. Finally, it is a pleasure to thank the editors and staff at Wiley, especially Debbie Jupe, Heather Kay, and Prachi Sinha Sahay. Thanks are also due to the project manager, Sangeetha Parthasarathy, for cheerfully and patiently overseeing the typesetting of the book.

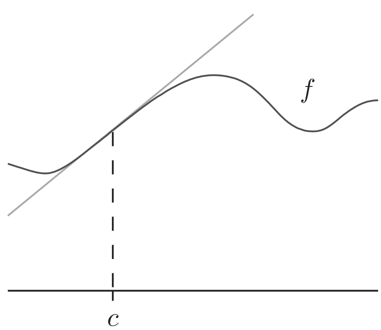
Amol Sasane
London, 2014.

Introduction

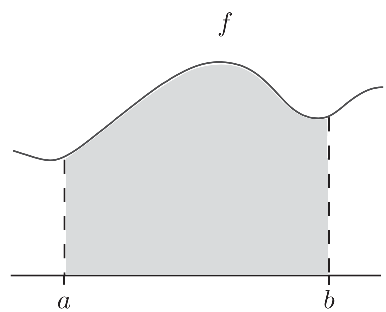
What is Calculus?

Calculus is a branch of mathematics in which the focus is on two main things: given a real-valued function of a real variable, what is the rate of change of the function at a point (Differentiation), and what is the area under the graph of the function over an interval (Integration).

Differentiation and **Integration**



What is the steepness/slope of f at the point c?



What is the area under the graph of f over an interval from a to b?

Differentiation	**Integration**
Differentiation is concerned with velocities, accelerations, curvatures, etc. These are rates of change of function values and are defined *locally*.	Integration is concerned with areas, volumes, average values, etc. These take into account the totality of function values, and are *not* defined locally.

We will see later on that the rate of change of f at c is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

and what matters is not what the function is doing *far away* from the point c, but rather the manner in which the function behaves in the *vicinity* of c. This is what we mean when we say that 'differentiation is a local concept'. On the other hand, we will learn that for nice functions, the area will be given by an expression that looks like

$$\int_a^b f(x)dx = \lim_{N \to \infty} \sum_{n=0}^{N} f\left(a + \frac{b-a}{N} n\right) \frac{b-a}{N},$$

and we see that in the above process, the values of the function over the entire interval from a to b *do matter*. In this sense integration is a 'non-local' or 'global' process.

Thus it seems that in Calculus, there are these two quite *different* topics of study. However there is a remarkable fact, known as that *Fundamental Theorem of Calculus*, which creates a bridge between these seemingly different worlds: it says, roughly speaking that the processes of differentiation and integration are inverses of each other:

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{and} \quad \frac{d}{dx} \int_a^x f(\xi)d\xi = f(x).$$

This interaction between differentiation and integration provides a powerful body of understanding and calculational technique, called 'Calculus'. Problems that would be otherwise computationally difficult can be solved mechanically using a few simple Calculus rules, and without the exertion of a great deal of penetrating thought.

Why study Calculus?

The reason why Calculus is a standard component of all scientific undergraduate education is because it is universally applicable in Physics, Engineering, Biology, Economics, and so on. Here are a few very simple examples:

(1) What is the escape velocity of a rocket on the surface of the Earth?

(2) If a hole of radius 1 cm is drilled along a diametrical axis in a solid sphere of radius 2 cm, then what is the volume of the body left over?

(3) If a strain of bacteria grows at a rate proportional to the amount present, and if the population doubles in an hour, then what is the population of bacteria at any time t?

(4) If the manufacturing cost of x lamps is given by $C(x) = 2700 - 100x$, and the revenue function is given by $R(x) = x - 0.03x^2$, then what is the number of lamps maximising the profit?

We will primarily be concerned with developing and understanding the tools of Calculus, but now and then in the exercises and examples chosen, we will consider a few toy models from various application areas to illustrate how the techniques of Calculus have universal applications.

What will we learn in this book?

This book is divided into six chapters, listed below.

(1) The real numbers.
(2) Sequences.
(3) Continuity.
(4) Differentiation.
(5) Integration.
(6) Series.

This covers the core component of a *single/one* variable Calculus course, where the basic object of study is a real-valued function of *one* real variable, and one studies the themes of differentiation and integration for such functions. On the other hand, in *multi/several* variable Calculus, the basic object of study is an $\mathbb{R}^m$-valued/vector valued function of several variables, and the themes of differentiation and integration for such functions. This book does not cover this latter subject.

We refrain from giving a brief gist of the contents of each of the chapters, since it won't make much sense to the novice at this stage, but instead we appeal to whatever previous exposure the student might have had in high school regarding these concepts. We will of course study each of these topics from scratch. We make one pertinent point though in the paragraph below.

A discussion of Calculus needs an ample supply of examples, which are typically through considering specific functions one meets in applications. The simplest among these illustrative functions are the algebraic functions, but it would be monotonous to just consider these. Much more interesting things happen with the so-called elementary transcendental functions such as the logarithm, exponential function, trigonometric functions, and so on. A rigorous definition of these unfortunately needs the very tools of Calculus that are being developed in this course. It would be a shame, however, if such rich examples centered around these functions have to wait till a rigorous treatment has been done. So we adopt a dual approach: we *will* choose to illustrate our definitions/theorems with these functions, and *not exclude* these functions from our preliminary discussion, hoping that the student has *some* exposure to the definitions (at whatever intuitive/rigour level) and properties of these transcendental functions. Later on, when the time is right (Chapter 5), we will give the precise mathematical definitions of these functions and prove the very properties that were accepted on faith in the initial parts of this book. This dual approach adopted by us has the advantage of not depriving the student of the nice illustrations of the results provided by these functions, and of preparing the student for the actual treatment of these functions later on. In any case, if the student meets a very unfamiliar property or manipulation involving these functions in the initial part of this book, it is safe to simply skip the relevant part and revisit it after Chapter 5 has been read.

How did Calculus arise?

Some preliminary ideas of Calculus are said to date back to as much as 2000 years ago when Archimedes determined areas using the Method of Exhaustion; see the following discussion and Figure 1.

The development of Differential and Integral Calculus is largely attributed to Newton (1642–1727) and Leibniz (1646–1716), and the foundations of the subject continued to be investigated into the 19th century, among others by Cauchy, Bolzano, Riemann, Weierstrass, Lebesgue, and so on.

We end this introduction with making a few remarks about the 'Method of Exhaustion', which besides treating this historical milestone in the development of Calculus, will also provide some motivation to begin our journey into Calculus with a study of the real numbers.

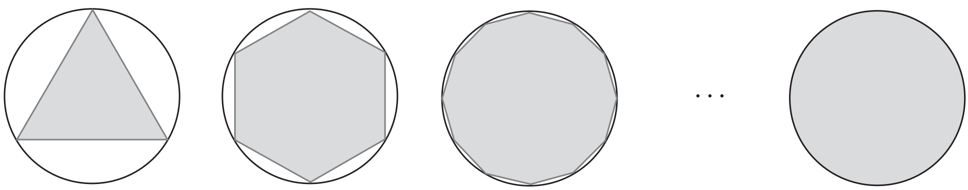


Figure 1. Determination of the area of a circle using the Method of Exhaustion.

In Figure 1, it is clear that what we are doing is trying to obtain the area of a circle by inscribing polygons inside it, each time doubling the number of sides, hence 'exhausting' more and more of the circular area. The idea is then that if A is the area of the circle we seek, and a_n is the area of the polygon at the nth step, then for large n, a_n approximates A. As we have that $a_1 \leq a_2 \leq a_3 \leq \cdots$, and since a_n misses A by smaller and smaller amounts as n increases, we expect that A should the 'smallest' number exceeding the numbers $a_1, a_2, a_3, \cdots$. Does such a number always exist?

Obviously, one can question the validity of this heuristic approach to solving the problem. The objections are for example:

(1) We did not really define what we mean by the area enclosed.

(2) We are not sure about what properties of numbers we are allowed to use. For example, we seem to be needing the fact that 'if we have an increasing sequence of numbers, all of which are less than a certain number[1], then there is a smallest number which is bigger than each of the numbers $a_1, a_2, a_3, \cdots$'. Is this property true for rational numbers?

Such questions might seem frivolous to a scientist who is just interested in 'real world applications'. But such a sloppy attitude can lead to trouble. Indeed, some work done in the 16th to the 18th century relying on a mixture of deductive reasoning and intuition, involving vaguely defined terms, was later shown to be *incorrect*. To give the student a quick example of how things might easily go wrong, one might naively, but incorrectly, guess that the answer to question (2) above is yes. This prompts the question of whether there is a bigger set of numbers than the rational numbers for which the property happens to be true? The answer is yes, and this is the **real number system** $\mathbb{R}$.

Thus a thorough treatment of Calculus must start with a careful study of the number system in which the action of Calculus takes place, and this is the real number system $\mathbb{R}$, where our journey begins!

[1] imagine a square circumscribing the circle: then each of the numbers $a_1, a_2, a_3, \cdots$ are all less than the area of the square

Preliminary notation

$A := B$ or $B =: A$	A is defined to be B; A is defined by B
$\forall$	for all; for every
$\exists$	there exists
$\neg\, S$	negation of the statement S; it is not the case that S
$a \in A$	the element a belongs to the set A
$\emptyset$	the empty set containing no elements
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a subset of B, but is not equal to B
$A \backslash B$	the set of elements of A that do not belong to B
$A \cap B$	intersection of the sets A and B
$\bigcap_{i \in I} A_i$	intersection of the sets A_i, $i \in I$
$A \cup B$	union of the sets A and B
$\bigcup_{i \in I} A_i$	union of the sets A_i, $i \in I$
$A_1 \times \cdots \times A_n$	Cartesian product of the sets $A_1, \cdots, A_n$; $\{(a_1, \cdots, a_n) : a_1 \in A_1, \cdots, a_n \in A_n\}$

1

The real numbers

From the considerations in the Introduction, it is clear that in order to have a firm foundation of Calculus, one needs to study the real numbers carefully. We will do this in this chapter. The plan is as follows:

(1) An intuitive, visual picture of $\mathbb{R}$: the number line. We will begin our understanding of $\mathbb{R}$ intuitively as points on the 'number line'. This way, we will have a mental picture of $\mathbb{R}$, in order to begin stating the precise properties of the real numbers that we will need in the sequel. It is a legitimate issue to worry about the actual construction of the set of real numbers, and we will say something about this in Section 1.8.

(2) Properties of $\mathbb{R}$. Having a rough feeling for the real numbers as being points of the real line, we will proceed to state the precise properties of the real numbers we will need. So we will think of $\mathbb{R}$ as an undefined set for now, and just state rigorously what properties we need this set $\mathbb{R}$ to have. These desirable properties fall under three categories:

(a) *the field axioms*, which tell us about what laws the arithmetic of the real numbers should follow,

(b) *the order axiom*, telling us that comparison of real numbers is possible with an order $>$ and what properties this order relation has, and

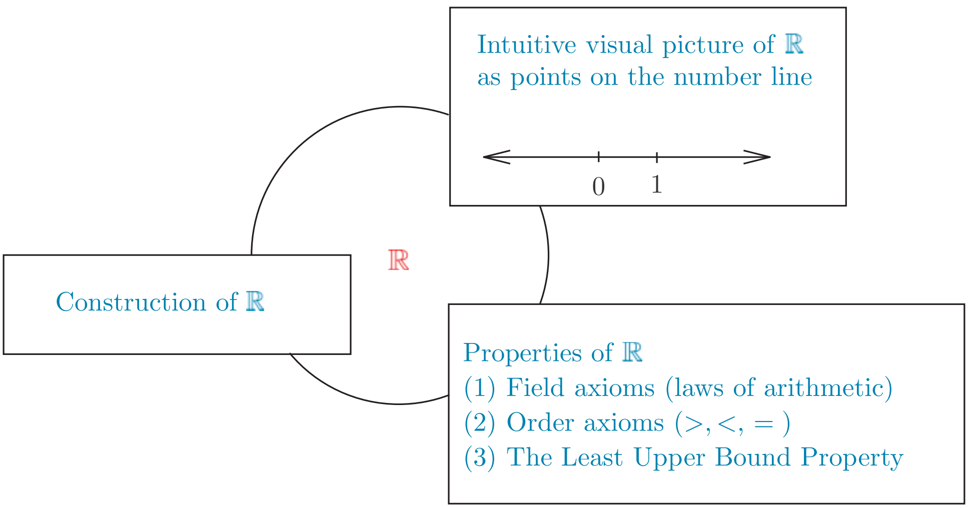
(c) *the Least Upper Bound Property of* $\mathbb{R}$, which tells us roughly that unlike the set of rational numbers, the real number line has 'no holes'. This last property is the most important one from the viewpoint of Calculus: it is the one which makes Calculus possible with real numbers. *If* rational numbers had this nice property, then we would not have bothered studying real numbers, and instead we would have just used rational numbers for doing Calculus.

The How and Why of One Variable Calculus, First Edition. Amol Sasane.

(3) The construction of $\mathbb{R}$. Although we will think of real numbers intuitively as 'numbers that can be depicted on the number line', this is not acceptable as a rigorous mathematical definition. So one can ask:

> Is there really a set $\mathbb{R}$ that can be constructed which has the stipulated properties (2)(a), (b), and (c) (and which will be detailed further in Sections 1.2, 1.3, 1.4)?

The answer is yes, and we will make some remarks about this in Section 1.8.



1.1 Intuitive picture of $\mathbb{R}$ as points on the number line

In elementary school, we learn about

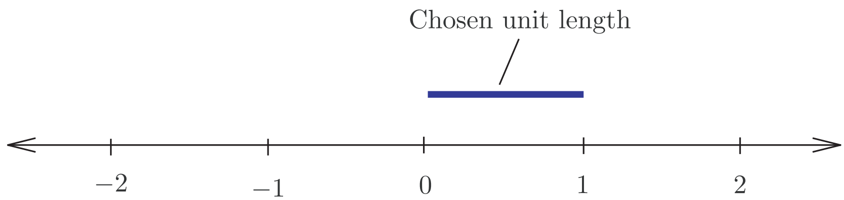
$$\begin{aligned} \text{the natural numbers} \quad & \mathbb{N} := \{1, 2, 3, \cdots\} \\ \text{the integers} \quad & \mathbb{Z} := \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}, \text{ and} \\ \text{the rational numbers} \quad & \mathbb{Q} := \left\{ \left[\frac{n}{d}\right] : n, d \in \mathbb{Z}, d \neq 0 \right\}. \end{aligned}$$

Incidentally, the rationale behind denoting the rational numbers by $\mathbb{Q}$ is that it reminds us of '**q**uotient', and $\mathbb{Z}$ for integers comes from the German word '**z**ählen' (meaning 'count'). In the above,

$$\left[\frac{n}{d}\right]$$

represents a whole family of 'equivalent fractions'; for example, $\frac{2}{4} = \frac{1}{2} = \frac{-3}{-6}$ etc.

We are accustomed to visualising these numbers on the 'number line'. What is the number line? It is any line in the plane, on which we have chosen a point O as the 'origin', representing the number 0, and chosen a unit length by marking off a point on the right of O, where the number 1 is placed. In this way, we get all the positive integers, $1, 2, 3, 4, \cdots$ by repeatedly marking off successively the unit length towards the right, and all the negative integers $-1, -2, -3, \cdots$ by repeatedly marking off successively the unit length towards the left.



Just like the integers can be depicted on the number line, we can also depict all rational numbers on it as follows. First of all, here is a procedure for dividing a unit length on the number line into d ($\in \mathbb{N}$) equal parts, allowing us to construct the rational number $1/d$ on the number line. See Figure 1.1.

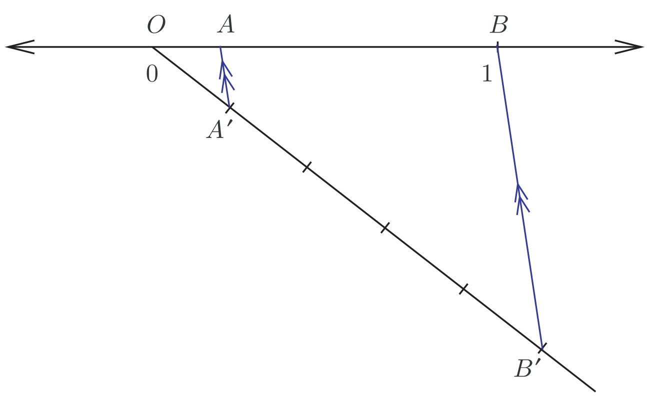


Figure 1.1 Construction of rational numbers: in the above picture, given the length 1 (that is, knowing the position of B), we can construct the length $1/5$, and so the point A corresponds to the rational number $1/5$.

The steps are as follows: Let the points O and B correspond to the numbers 0 and 1.

(1) Take any arbitrary length $\ell(OA')$ along a ray starting at O in any direction other than that of the number line itself.

(2) Let B' be a point on the ray such that $\ell(OB') = d \cdot \ell(OA')$.

(3) Draw AA' parallel to BB' to meet the number line at A.

Conclusion: From the similar triangles $\Delta OAA'$ and $\Delta OBB'$, we see that the length $\ell(OA) = 1/d$.

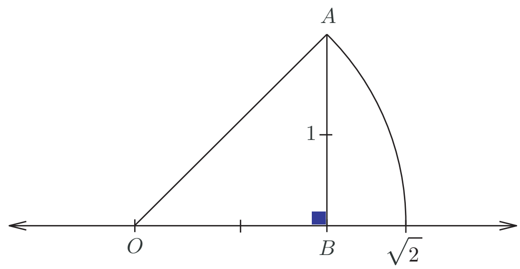
Having obtained $1/d$, we can now construct n/d on the number line for *any* $n \in \mathbb{Z}$, by repeating the length $1/d$ n times towards the right of 0 if $n > 0$, and towards the left $-n$ times from 0 if n is negative.

Hence, we can depict all the rational numbers on the number line. Does this exhaust the number line? That is, suppose that we start with all the points on the number line being coloured black, and suppose that at a later time, we colour all the rational ones by red: are there any black points left over? The answer is yes, and we demonstrate this below. We will show that there does 'exist', based on geometric reasoning, a point on the number line, whose square is 2, but we will also argue that this number, denoted by $\sqrt{2}$, is not a rational number.

First of all, the picture below shows that $\sqrt{2}$ exists as a point on the number line. Indeed, by looking at the right angled triangle ΔOBA, Pythagoras's Theorem tells us that the length of the hypotenuse OA satisfies

$$(\ell(OA))^2 = (\ell(OB))^2 + (\ell(AB))^2 = 1^2 + 1^2 = 2,$$

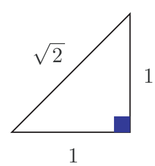
and so $\ell(OA)$ is a number, denoted say by $\sqrt{2}$, whose square is 2. By taking O as the centre and radius $\ell(OA)$, we can draw a circle using a compass that intersects the number line at a point C, corresponding to the number $\sqrt{2}$. Is $\sqrt{2}$ a rational number? We show below that it isn't!



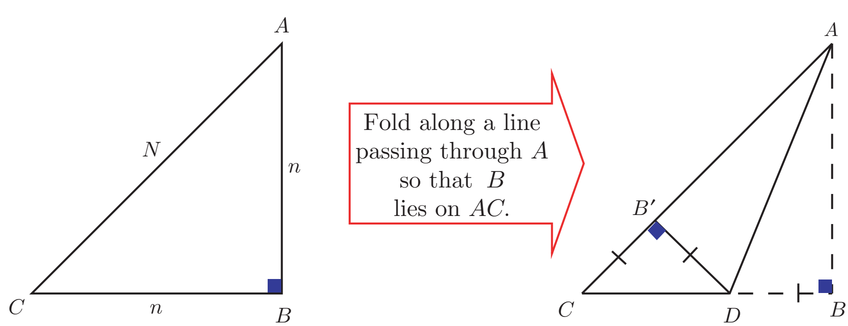
Exercise 1.1. Depict $-11/6$ and $\sqrt{3}$ on the number line.

Theorem 1.1 (An 'origami' proof of the irrationality of $\sqrt{2}$). *There is no rational number* $q \in \mathbb{Q}$ *such that* $q^2 = 2$.

Proof. Suppose that $\sqrt{2}$ is a rational number. Then some scaling of the triangle



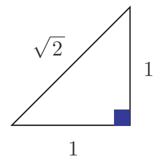
by an integer will produce a similar triangle, all of whose sides are integers. Choose the smallest such triangle, say ΔABC, with integer lengths $\ell(BC) = \ell(AB) = n$, and $\ell(AC) = N$, $n, N \in \mathbb{N}$. Now do the following origami: fold along a line passing through A so that B lies on AC, giving rise to the point B' on AC. The 'crease' in the paper is actually the angle bisector AD of the angle $\angle BAC$.



In $\Delta CB'D$, $\angle CB'D = 90°$, $\angle B'CD = 45°$. So $\Delta CB'D$ is an isosceles right triangle. We have $\ell(CB') = \ell(B'D) = \ell(AC) - \ell(AB') = N - n \in \mathbb{N}$, while

$$\ell(CD) = \ell(CB) - \ell(DB) = n - \ell(B'C) = n - (N - n) = 2n - N \in \mathbb{N}.$$

So $\Delta CB'D$ is similar to the triangle



has integer side lengths, and is smaller than ΔABC, contradicting the choice of ΔABC. So there is no rational number q such that $q^2 = 2$. □

A different proof is given in the exercise below.

Exercise 1.2. (∗) We offer a different proof of the irrationality of $\sqrt{2}$, and en route learn a technique to prove the irrationality of 'surds'.[1]

(1) Prove the Rational Zeros Theorem: Let $c_0, c_1, \cdots, c_d$ be $d \geq 1$ integers such that c_0 and c_d are not zero. Let $r = p/q$ where p, q are integers having no common factor and such that $q > 0$. Suppose that r is a zero of the polynomial $c_0 + c_1 x + \cdots + c_d x^d$. Then q divides c_d and p divides c_0.
(2) Show that $\sqrt{2}$ is irrational.
(3) Show that $\sqrt[3]{6}$ is irrational.

Thus, we have seen that the elements of $\mathbb{Q}$ can be depicted on the number line, and that not all the points on the number line belong to $\mathbb{Q}$. We think of $\mathbb{R}$ as *all* the points on the number line. As mentioned earlier, if we take out everything on the number line (the black points) except for the rational numbers $\mathbb{Q}$ (the red points), then there will be holes among the rational numbers (for example, there will be a missing black point where $\sqrt{2}$ lies on the number line). We can think of the real numbers as 'filling in' these holes between the rational numbers. We will say more about this when we make remarks about the construction of $\mathbb{R}$. Right now, we just have an intuitive picture of the set of real numbers as a bigger set than the rational numbers, and we think of the real numbers as points on the number line. Admittedly, this is certainly not a mathematical definition, and is extremely vague. In order to be precise, and to do Calculus rigorously, we just can't rely on this vague intuitive picture of the real numbers. So we now turn to the precise properties of the real numbers that we are allowed to use in

[1] Surds refer to irrational numbers that arise as the nth root of a natural number. The mathematician al-Khwarizmi (around 820 AD) called irrational numbers 'inaudible', which was later translated to the Latin *surdus* for 'mute'.

developing Calculus. While stating these properties, we will think of the set $\mathbb{R}$ as an (as yet) undefined set containing $\mathbb{Q}$ which will satisfy the properties of

(1) the field axioms (laws of arithmetic in $\mathbb{R}$),

(2) the order axioms (allowing us to compare real numbers with $>, <, =$), and

(3) the Least Upper Bound Property (making Calculus possible in $\mathbb{R}$),

stipulated below.

It is a pertinent question if one can construct (if there really exists) such a set $\mathbb{R}$ satisfying the above properties (1–3). The answer to this question is yes, but it is tedious. So in this first introductory course, we will not worry ourselves too much with it. It is a bit like the process of learning physics: typically one does not start with quantum mechanics and the structure of an atom, but with the familiar realm of classical mechanics. To consider another example, imagine how difficult it would be to learn a foreign language if one starts to painfully memorise systematically all the rules of grammar first; instead a much more fruitful method is to start practicing simple phrases, moving on to perhaps children's comic books, listening to pop music in that language, news, literature, and so on. Of course, along the way one picks up grammar and a formal study can be done at leisure later resulting in better comprehension. We will actually give some idea about the construction of the real numbers in Section 1.8. Right now, we just accept on faith that the construction of $\mathbb{R}$ possessing the properties we are about to learn can be done, and to have a concrete object in mind, we rely on our familiarity with the number line to think of the real numbers when we study the properties (1), (2), (3) listed above.

We also remark that property (3) (the Least Upper Bound Property) of $\mathbb{R}$ will turn out to be crucial for doing Calculus. The properties (1), (2) are also possessed by the rational number system $\mathbb{Q}$, but we will see that (3) fails for $\mathbb{Q}$.

1.2 The field axioms

The content of this section can be summarised in one sentence: $(\mathbb{R}, +, \cdot)$ forms a field. What does this mean? It is a compact way of saying the following. $\mathbb{R}$ is a set, equipped with two operations:

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

called *addition*, which sends a pair of real numbers (x, y) to their *sum* $x + y$, and the other operation is

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

called *multiplication*, which sends a pair of real numbers (x, y) to their *product* $x \cdot y$, and these two operations satisfy certain laws, called the 'field axioms'.[2] The field axioms for $\mathbb{R}$ are

[2] There are other number systems, for example, the rational numbers $\mathbb{Q}$ which also obey similar laws of arithmetic, and so $(\mathbb{Q}, +, \cdot)$ is also deemed to be a field. So the word 'field' is invented to describe the situation that one has a number system $\mathbb{F}$ with corresponding operations $+$ and $\cdot$ which obey the usual laws of arithmetic, rather than listing all of these laws.

listed below:

$$+\begin{cases} \text{(F1)} & \text{(Associativity)} & \text{For all } x, y, z \in \mathbb{R},\ x + (y + z) = (x + y) + z. \\ \text{(F2)} & \text{(Additive identity)} & \text{For all } x \in \mathbb{R},\ x + 0 = x = 0 + x. \\ \text{(F3)} & \text{(Inverses)} & \text{For all } x \in \mathbb{R}, \text{there exists } -x \in \mathbb{R} \\ & & \quad \text{such that } x + (-x) = 0 = -x + x. \\ \text{(F4)} & \text{(Commutativity)} & \text{For all } x, y \in \mathbb{R},\ x + y = y + x. \end{cases}$$

$$\cdot\begin{cases} \text{(F5)} & \text{(Associativity)} & \text{For all } x, y, z \in \mathbb{R},\ x \cdot (y \cdot z) = (x \cdot y) \cdot z. \\ \text{(F6)} & \text{(Multiplicative identity)} & 1 \neq 0 \text{ and for all } x \in \mathbb{R},\ x \cdot 1 = x = 1 \cdot x. \\ \text{(F7)} & \text{(Inverses)} & \text{For all } x \in \mathbb{R}\backslash\{0\},\ \text{there exists } x^{-1} \in \mathbb{R} \\ & & \quad \text{such that } x \cdot x^{-1} = 1 = x^{-1} \cdot x. \\ \text{(F8)} & \text{(Commutativity)} & \text{For all } x, y \in \mathbb{R},\ x \cdot y = y \cdot x. \end{cases}$$

$$+, \cdot \begin{cases} \text{(F9)} & \text{(Distributivity)} & \text{For all } x, y, z \in \mathbb{R},\ x \cdot (y + z) = x \cdot y + x \cdot z. \end{cases}$$

With these axioms, it is possible to prove the usual arithmetic manipulations we are accustomed to. Here are a couple of examples.

Example 1.1. For every $a \in \mathbb{R}$, $a \cdot 0 = 0$.

Let $a \in \mathbb{R}$. Then we have $a \cdot 0 \overset{F2}{=} a \cdot (0 + 0) \overset{F9}{=} a \cdot 0 + a \cdot 0$. So with $x := a \cdot 0$, we have got $x + x = x$. Adding $-x$ on both sides (F3!), and using (F1) we obtain

$$0 = x + (-x) = (x + x) + (-x) \overset{F1}{=} x + (x + (-x)) \overset{F3}{=} x + 0 \overset{F2}{=} x = a \cdot 0,$$

completing the proof of the claim. ♢

Example 1.2. If $a, b \in \mathbb{R}$, and $a \cdot b = 0$, then $a = 0$ or $b = 0$.

If $a = 0$, then we are done. Suppose that $a \neq 0$. By (F7), there exists a real number a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Hence

$$b = 1 \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0.$$

So if $a \neq 0$, then $b = 0$. Thus $\Big(a, b \in \mathbb{R} \text{ such that } a \cdot b = 0\Big) \Rightarrow \Big(a = 0 \text{ or } b = 0\Big)$. ♢

Of course in this book, we will not do such careful justifications every time we need to manipulate real numbers. We have listed the above laws to once and for all stipulate the laws of arithmetic for real numbers that justify the usual calculational rules we are familiar with, so that we know the *source* of it all. For example, the student may wish to try his/her hand at producing a rigorous justification based on (F1) to (F9) of the following well known facts.

Exercise 1.3. $(*)$ Using the field axioms of $\mathbb{R}$, prove the following:

(1) Additive inverses are unique.

(2) For all $a \in \mathbb{R}$, $(-1) \cdot a = -a$.

(3) $(-1) \cdot (-1) = 1$.

1.3 Order axioms

We now turn to order axioms for the real numbers. This is the source of the inequality '$>$' that we are used to, enabling one to compare two real numbers. The relation $>$ between real numbers arises from a special subset $\mathbb{P}$ of the real numbers.

Order axiom. There exists a subset $\mathbb{P}$ of $\mathbb{R}$ such that

(O1) If $x, y \in \mathbb{P}$, then $x + y \in \mathbb{P}$ and $x \cdot y \in \mathbb{P}$.

(O2) For every $x \in \mathbb{R}$, *one and only one* of the following statements is true:

$\underline{1}^\circ$ $x = 0$.
$\underline{2}^\circ$ $x \in \mathbb{P}$.
$\underline{3}^\circ$ $-x \in \mathbb{P}$.

Definition 1.1 (Positive numbers). The elements of $\mathbb{P}$ are called *positive numbers*. For real numbers x, y, we say that

$x > y$ if $x - y \in \mathbb{P}$,
$x < y$ if $y - x \in \mathbb{P}$,
$x \geq y$ if $x = y$ or $x > y$,
$x \leq y$ if $x = y$ or $x < y$.

It is clear from (O2) that 0 is *not* a positive number. Also, from (O2) it follows that for real numbers x, y, *one and only one* of the following statements is true:

$\underline{1}^\circ$ $x = y$.

$\underline{2}^\circ$ $x > y$.

$\underline{3}^\circ$ $x < y$.

Why is this so? If $x \neq y$, then $x - y \neq 0$, and so by (O2), we have the mutually exclusive possibilities $x - y \in \mathbb{P}$ or $y - x = -(x - y) \in \mathbb{P}$ happening, that is, either $x > y$ or $x < y$.

Example 1.3. $1 > 0$.

We have three possible, mutually exclusive cases:

$\underline{1}^\circ$ $1 = 0$.

$\underline{2}^\circ$ $1 \in \mathbb{P}$.

$\underline{3}^\circ$ $-1 \in \mathbb{P}$.

As $1 \neq 0$, we know that 1° is not possible.

Suppose that 3° holds, that is, $-1 \in \mathbb{P}$. From Exercise 1.3(3), $(-1) \cdot (-1) = 1$. Using (O1), and the fact that $-1 \in \mathbb{P}$, it then follows that $1 = (-1) \cdot (-1) \in \mathbb{P}$. So if we assume that 3° holds, then we obtain that *both* 2° and 3° are true, which is impossible as it violates (O2).

Thus by (O2), the only remaining case, namely 2° must hold, that is, $1 \in \mathbb{P}$. ♢

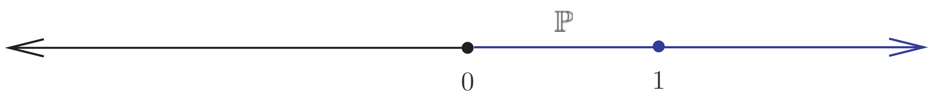
Exercise 1.4. ($*$) Using the order axioms for $\mathbb{R}$, show the following:

(1) For all $a \in \mathbb{R}$, $a^2 \geq 0$.

(2) There is no real number x such that $x^2 + 1 = 0$.

Again, just like we can use the field axioms to justify arithmetic manipulations of real numbers, it is enough to know that if challenged, one can derive all the usual laws of manipulating inequalities among real numbers based on these order axioms, but we will not do this at every instance we meet an inequality.

From our intuitive picture of $\mathbb{R}$ as points on the number line, what is the set $\mathbb{P}$? $\mathbb{P}$ is simply the set of all points/real numbers to the right of the origin O.



Also, geometrically on the number line, the inequality $a < b$ between real numbers a, b means that b lies to the right of a on the number line.

1.4 The Least Upper Bound Property of $\mathbb{R}$

This property is crucial for proving the results of Calculus, and when studying the proofs of the key results (the Bolzano–Weierstrass Theorem, the Intermediate Value Theorem, the Extreme Value Theorem, and so on), we will gradually learn to appreciate the key role played by it.

Definition 1.2 (Upper bound of a set). Let S be a subset of $\mathbb{R}$. A real number u is said to be an *upper bound of* S if for all $x \in S$, $x \leq u$.

If we think of the set S as some blob on the number line, then u should be any point on the number line that lies to the right of the points of the blob.

Example 1.4.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then 1976 is an upper bound of S. In fact, any real number $u \geq 1976$ is an upper bound of S. So S has lots of upper bounds.

(2) Let $S := \{x \in \mathbb{R} : x < 1\}$. Then 1 is an upper bound of S. In fact, any real number $u \geq 1$ is an upper bound of S.

(3) If $S = \mathbb{R}$, then S has no upper bound. Why? Suppose that $u \in \mathbb{R}$ is an upper bound of $\mathbb{R}$. Consider $u + 1 \in S = \mathbb{R}$. Then

$$\underbrace{u+1}_{\in S} \leq \underbrace{u}_{\text{upper bound of } S},$$

and so $1 \leq 0$, a contradiction!

(4) Let $S = \emptyset$ (the empty set, containing no elements). *Every* $u \in \mathbb{R}$ is an upper bound. For if $u \in \mathbb{R}$ is not an upper bound of S, then there must exist an element $x \in S$ which prevents u from being an upper bound of S, that is,

$$\boxed{\text{it is not the case that } x \leq u}$$

But S has no elements at all, much less an element such that $\boxed{\cdots}$ holds.

(This is an example of a 'vacuous truth'. Consider the statement

$$\boxed{\text{Every man with 60000 legs is intelligent.}}$$

This is considered a true statement in Mathematics. The argument is: Can you show me a man with 60000 legs for which the claimed property (namely of being intelligent) is not true? No! Because there are no men with 60000 legs! By the same logic, even the statement

$$\boxed{\text{Every man with 60000 legs is not intelligent.}}$$

is true in Mathematics.) ◇

Definition 1.3 (Set bounded above). If $S \subset \mathbb{R}$ and S has an upper bound (that is, the set of upper bounds of S is not empty), then S is said to be *bounded above*.

Example 1.5. The set $\mathbb{R}$ is not bounded above.

Each of the sets $\{0, 1, 9, 7, 6, 1976\}$, $\emptyset$, $\{x \in \mathbb{R} : x < 1\}$ is bounded above. ◇

Similarly one can define the notions of a lower bound, and of a set being bounded below.

Definition 1.4 (Lower bound of a set; set bounded below). Let S be a subset of $\mathbb{R}$. A real number ℓ is said to be a *lower bound of* S if for all $x \in S$, $\ell \leq x$.

If $S \subset \mathbb{R}$ and S has a lower bound (that is, the set of lower bounds of S is not empty), then S is said to be *bounded below*.

If we think of the set S as some blob on the number line, then ℓ should be any point on the number line that lies to the left of the points of the blob.

Example 1.6.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then 0 is a lower bound of S. In fact, any real number $\ell \le 0$ serves as a lower bound of S. So S is bounded below.

(2) Let $S := \{x \in \mathbb{R} : x < 1\}$. Then S is not bounded below. Let us show this. Suppose that, on the contrary, S does have a lower bound, say $\ell \in \mathbb{R}$. Let $x \in S$. Then $\ell \le x < 1$. We have

$$\ell - 1 < \ell \le x < 1,$$

and so $\ell - 1 < 1$. Thus $\ell - 1 \in S$, and as ℓ is a lower bound of S, we must have $\ell \le \ell - 1$, that is, $1 < 0$, a contradiction! So our original assumption that S is bounded below must be false. Thus S is not bounded below. (This claim was intuitively obvious too, since the set of points in S on the number line is the entire ray of points on the left of 1, leaving no room for points on $\mathbb{R}$ to be on the 'left of S'.)

(3) If $S = \mathbb{R}$, then S has no lower bound. Indeed, if $\ell \in \mathbb{R}$ is a lower bound of $\mathbb{R}$, then

$$\underbrace{\ell}_{\text{lower bound of } S} \le \underbrace{\ell - 1}_{\in S},$$

and so $1 \le 0$, a contradiction. Thus $\mathbb{R}$ is not bounded below.

(4) Let $S = \emptyset$ (the empty set, containing no elements). Every $\ell \in \mathbb{R}$ is a lower bound. If $\ell \in \mathbb{R}$ is not a lower bound of S, then there must exist an element $x \in S$ which prevents ℓ from being a lower bound of S, that is, it is not the case that $\ell \le x$. But as S is empty, this is impossible. So S is bounded below. ♢

Definition 1.5 (Bounded set). Let $S \subset \mathbb{R}$. S is called *bounded* if S is bounded below and bounded above.

Example 1.7.

S	An upper bound	Bounded above?	A lower bound	Bounded below?	Bounded?
$\{0, 1, 9, 7, 6, 1976\}$	1976 Any $u \ge 1976$	Yes	0 Any $\ell \le 0$	Yes	Yes
$\{x \in \mathbb{R} : x < 1\}$	1 Any $u \ge 1$	Yes	Does not exist	No	No
$\mathbb{R}$	Does not exist	No	Does not exist	No	No
$\emptyset$	Every $u \in \mathbb{R}$	Yes	Every $\ell \in \mathbb{R}$	Yes	Yes

♢

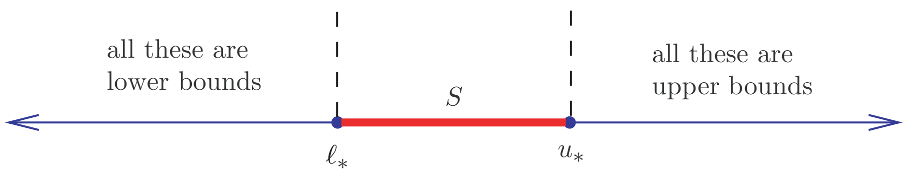
We now introduce the notions of a least upper bound (also called supremum) and a greatest lower bound (also called infimum) of a subset S of $\mathbb{R}$.

Definition 1.6 (Supremum and infimum). Let S be a subset of $\mathbb{R}$.

(1) $u_* \in \mathbb{R}$ is called a *least upper bound of S* (or a *supremum of S*) if
   (a) u_* is an upper bound of S, and
   (b) if u is an upper bound of S, then $u_* \leq u$.

(2) $\ell_* \in \mathbb{R}$ is called a *greatest lower bound of S* (or an *infimum of S*) if
   (a) ℓ_* is a lower bound of S, and
   (b) if ℓ is a lower bound of S, then $\ell \leq \ell_*$.

Pictorially, the supremum is the leftmost point among the upper bounds, and the infimum is the rightmost point among the lower bounds of a set.

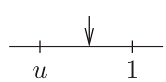


Example 1.8.

(1) If $S = \{0, 1, 9, 7, 6, 1976\}$, then $u_* = 1976$ is a least upper bound of S because
   (1) 1976 is an upper bound of S, and
   (2) if u is an upper bound of S, then $(S \ni)$ $1976 \leq u$, that is $u_* \leq u$.

Similarly, 0 is a greatest lower bound of S.

(2) Let $S = \{x \in \mathbb{R} : x < 1\}$. Then we claim that $u_* = 1$ is a least upper bound of S. Indeed we have:
   (a) 1 is an upper bound of S: If $x \in S$, then $x < 1 = u_*$.
   (b) Let u be an upper bound of S. We want to show that $u_* = 1 \leq u$. Suppose the contrary, that is, $1 > u$. Then there is a gap between u and 1.



(But then we see that this gap between u and 1 of course contains elements of S which are to *right* of the supposed upper bound u, and this should give the contradiction we seek.) To this end, let us consider the number $(1 + u)/2$. We have

$$\frac{1+u}{2} < \frac{1+1}{2} = 1$$