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The Geometry of Special Relativity—a Concise Course

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Preface

So einfach wie möglich, aber nicht einfacher

As simple as possible, but not simpler. This guideline of Albert Einstein obliges in particular each presentation of relativistic physics, a subject which often puzzles laymen, stirs their imagination, and tantalizes their comprehension, unnecessarily, because relativistic physics relies on simple geometric notions.

If one wants to understand the basic features of the theory of relativity then one does not need coordinates or virtual systems of clocks, which fill the universe, no more than millimeter paper and coordinate axes are required for Euclidean geometry. One only has to consider what observers see rather than to argue that this or that observer is right. Relativity is a physical, not a judicial theory.

The slowdown of moving clocks and the shortening of a moving measuring rod unfold naturally from the principle of relativity, just as a tilted ladder is less high than an upright ladder. Clocks are no more mysterious than mileage meters, and show a distance between start and end which depends on the way in between. This is the unspectacular answer to the seemingly paradoxical aging of twins. Just as no one is puzzled by a triangle, where the straight line between two edges is shorter than the detour over the third edge, no one should be shocked by the conclusion and experimental verification, that a clock picks up more time on a straight history as compared to the twin clock of a traveler who takes a detour.

The first two chapters are intended to be understandable in essence also to non-physicists with little mathematical knowledge. Their simplicity, however, may be deceptive. Real understanding requires careful consideration of the arguments, the equations, and the diagrams, preferably by reading equipped with a pencil and paper.

The following chapters presume mathematical knowledge which physicists and mathematicians acquire during their undergraduate years. To clarify more complicated questions we introduce coordinates as functions of the measured times and directions of light rays and deduce the Lorentz transformations which relate these values to the ones which moving observers measure. These transformations determine how velocities combine, what pictures are seen by moving observers, and how the energy and momentum of a particle depend on its velocity.

Chapter 4 assembles the basics of mechanics and applies them to relativistic particles. Stress is laid on the correspondence between physics and geometry, between conserved quantities like energy, momentum and angular momentum, and symmetries like a shift in time or space or a rotation or a Lorentz transformation. Jet spaces, which are introduced and used in this investigation, may strike the reader as an unnecessary complication. But they provide the clearest and therefore simplest setting to exhibit the correspondence of conserved quantities and infinitesimal symmetries.

Chapter 5 presents electrodynamics as a relativistic field theory and in particular shows that changes of the electric charges cause changes of the electromagnetic fields with the speed of light. The electrodynamic interactions are invariant under dilations, which is why they cannot explain the particular values of particle masses or the particular sizes of atoms.

In the last chapter we discuss the mathematical properties of the Lorentz group. It acts on the directions of light rays just as the Möbius transformations act on the Riemann sphere.

The text originated from courses which I taught on the subject and from my answers to questions which were frequently asked in the newsgroup de.sci.physik. After a few years the notes changed nearly no more and slumbered on my homepage with a few hundred interested visitors per year until Christian Caron from Springer Verlag encouraged me to have them published. Whether this kiss of a prince awoke a sleeping beauty or a frog, still to be thrown against the wall, is the reader to judge.

Helpful comments and patient listening were contributed by Frédéric Arenou, Werner Benger, Christian Böhmer, Christoph Dehne, Jürgen Ehlers, Christopher Eltschka, Chris Hillman, Olaf Lechtenfeld, Volker Perlick, Markus Pössel, and Bernd Schmidt. Ulrich Theis translated the early versions of the notes. Sincere thanks are given to Ulla and Hermann Nicolai for their friendly hospitality during my stay at the Albert-Einstein-Institut der Max-Planck-Gesellschaft.

Hannover, Germany, January 2012 Norbert Dragon

Contents

Chapter 1
Structures of Spacetime

Abstract Simple geometric properties of spacetime and free particles underlie the theory of relativity just as Euclidean geometry follows from simple properties of points and straight lines. The vacuum, the empty four-dimensional curved spacetime, determines straight lines and light rays. In the absence of gravity, the vacuum is isotropic and homogeneous and does not allow to distinguish rest from uniform motion. Therefore, contrary to Newton's opinion, the vacuum cannot contain the information about a universal time which could be attributed to events. Whether two different events are simultaneous depends on the observer—just as in Euclidean geometry it depends on a given direction whether two points lie on an orthogonal line.

1.1 Properties of the Vacuum

We can denote a point in space by specifying how far away it is ahead, to the right and to the top of a chosen reference point. These specifications are called coordinates of the point. One needs three coordinates in order to specify any one point. Space is three-dimensional. The coordinates of a point depend of course on the choice of the reference point and on which directions the observer chooses as ahead, right and above.

As for appointments in daily life, for physical processes not only the position is important, where an event takes place, but also the time when it occurs. The set of all events, spacetime, is four-dimensional, because to specify a single event one needs four labels, the position where it takes place and the time when it occurs. The position and time specifications which label an event depend—just as the three coordinates of a position—on the observer.

The four-dimensional spacetime fascinates and beats our imagination which is trained in everyday life. Nevertheless it is quite simple. We can easily envisage a stack of pictures, as they are stored in a film reel, which show the sequel of three-dimensional situations. Thereby one conceives the four-dimensional space-

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time the same way as an architect, who draws two-dimensional blueprints, horizontal plans and transversal sections to envisage a three-dimensional building.

Using the same means we depict the sequel of events in two-dimensional spacetime diagrams. For example, the geometric figure of two intersecting straight lines shows the physical process that two particles move uniformly and collide in the event, where the lines intersect. If one would display only the position and not the time one would not know whether the particles pass the same position at the same time or fail to meet each other.

The physical findings add the insight, which is alien to our intuition, that the four-dimensional spacetime is an entity which is decomposed into layers of equal time only by the observer. Different from what Newton thought, these slices of equal time do not coincide for observers which move relative to each other.

That simultaneity depends on the observer is the largest obstacle for understanding relativity. Not only the three coordinates of the position, but also the time which denotes an event, depend on the observer. Spacetime has no measurable universal time which pertains to the events.

Each event E determines the later events which can be influenced by E by means of light or electromagnetic signals, and vice versa, it determines the earlier events from which it could have been influenced. These events form the forward and backward lightcone of E, which both belong geometrically to each event in spacetime.

An unaccelerated clock which passes two events shows the time which passes in between. This time does not depend on the particular type of clock and is a geometrical property of the two events, their temporal distance. From this time all length standards derive, in particular spatial distance is the time which it takes light to run back and forth (1.5). The temporal distance of events imparts a geometry to spacetime which is similar to Euclidean geometry in many respects. As we will see, the events with the same temporal distance from a chosen event lie on a hyperboloid and not, as in Euclidean geometry, on a sphere. The geometry of spacetime is ordinary school geometry, with circles replaced by hyperbolas.

To avoid the multitude of possible effects, we investigate processes in an empty region of spacetime, the vacuum, from which all particles have been removed and all influences from outside, such as electric and magnetic fields, are shielded. This vacuum is the stage on which we study the behavior of light and particles that are seen by observers and are measured with clocks and measuring rods.

As simple as the idea of a vacuum seems, it is an idealization and can be realized only approximately. We are constantly passed by neutrinos, which come from the sun and single out a particular direction in the otherwise isotropic space. We cannot shield our experiments from these neutrinos since they do not interact sufficiently. But since neutrinos penetrate everything, they do not bother either and cause effects only if we look for them on purpose.

The cosmos filled by background radiation is not a vacuum. This radiation is a remnant from the early evolution of the universe and defines a rest frame through which the sun moves with a speed of roughly 370 km/s [28]. This background radiation can be shielded by walls, but the walls have to be cooled so that the heat radiation of the walls does not fill the space.

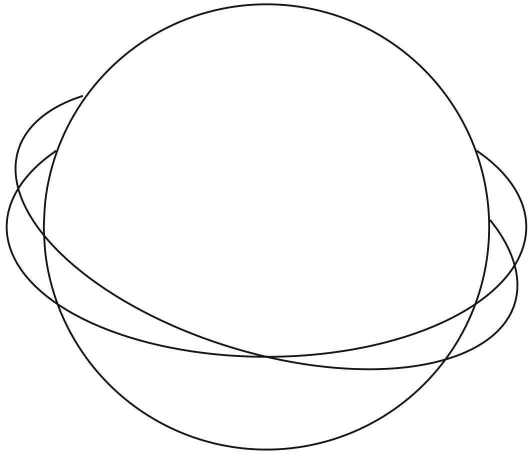

Fig. 1.1 Orbits around the Earth

Omnipresence of Gravity

Even after removing all particles and shielding from all external influences the vacuum retains structure.

Gravity cannot be shielded or extracted from the vacuum. It therefore belongs to the properties of the empty spacetime, though in special arrangements one can compensate gravitational attraction in a region by additional masses. For example, if one completes a segment of a spherical shell, which causes gravitational attraction, to a complete shell, then in the interior of the shell the gravitation of the segment is compensated. However, one cannot compensate unforeseen disturbances from outside by a Faraday cage with particles, which can move freely in a wire with negligible inertia. For freely moving particles, the state with lowest energy is not a vanishing gravitational field, but, because gravity is attractive, the closest packing of the particles and the largest possible gravitational attraction. They do not shield gravity but enlarge it. Also, there are no bodies which are inert and nearly insensitive to gravity. Irrespective of their masses all test particles fall in the same way.

One cannot completely transform away gravity by performing experiments in freely falling laboratories. In a laboratory which orbits the earth in free fall one can distinguish three different directions, behind, below and beneath, by gravitational effects without view to the earth and without reference to the walls. Test particles in the laboratory orbit the earth in ellipses, in the simplest case in circles. If the test particles orbit circles in the same plane with different radii, then the particle which is nearer to earth is faster and departs from the other. If the test particles cycle behind each other in the same circle, then their distance remains unchanged. If they orbit initially beneath each other in different circles of the same radius, then the orbital planes intersect and the circles intersect twice per revolution: freely falling particles beneath each other oscillate around each other with the orbital frequency.

Straight Worldlines in Curved Spacetime

In the vacuum an observer can determine without reference to other positions whether he is freely falling or accelerating. This he can read off from an hour glass—it does not run in free fall—or from a pendulum, which he carries along. If it swings back and forth, an acceleration acts vertically to the axis of rotation, otherwise the pendulum rotates with constant angular velocity.

An observer passes in the course of time a set of events. This line in spacetime is his worldline. For each freely falling particle this line is determined by an event, which it passes and by the velocity in that instant, corresponding to a point and a direction with which the worldline traverses the point. These worldlines do not depend on particulars of the particles, because all particles fall the same way. Therefore the worldlines of freely falling particles define a geometrical structure of spacetime itself: the set of straight lines.

Note well: the worldlines of freely falling particles and of flashes of light are the straight lines of the four-dimensional spacetime, but their spatial projection are not straight lines of the three-dimensional space. Through each point in space and with given direction there pass different parabolas which are traced out by falling particles with different velocities. These curves in three-dimensional space are not determined—differently from what one has to require from straight lines—by a point and a direction.

Among the spatial curves of freely falling particles one can and does choose a class of curves to define straight lines, if gravity does not change with time. Straight is the path of light. Whether an edge is straight is checked by comparing with light, by sighting along the edge. But light rays are gravitationally deflected and can intersect each other repeatedly. They do not satisfy Euclid's axiom of parallels. They define straight lines in a space which is curved by gravity.

If gravity changes in time because the masses move which generate gravity, then the light rays do no longer define straight spatial lines, because the ways to and fro differ.

One could imagine to attribute the label straight to other lines in spacetime, for example to lines which in some coordinate system can be drawn with a ruler. But these lines are no property of spacetime and test particles traverse them only if they are subject to forces which differ for different particles. The only worldlines which are singled out by nature are the worldlines of freely falling particles, including the worldlines of flashes of light.

If one follows mentally the path of freely falling particles, which in Fig. 1.1 orbit the earth in different circles, then one realizes that spacetime is curved. If one relates the positions of the second particle to the positions of the first particle, then the straight worldline of the second particle oscillates around the straight worldline of the first particle, straight lines can intersect each other repeatedly.

The reason for the curvature of spacetime and the relative motion of freely falling particles is the fact, that gravity is not the same everywhere: the attraction is stronger the nearer the particles and in equal distance it acts in different directions. By the

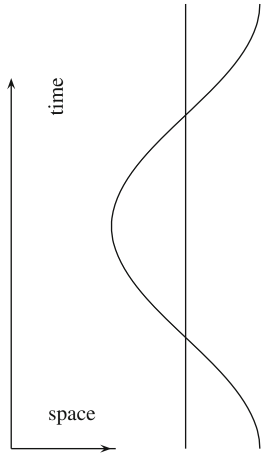

Fig. 1.2 Straight lines in curved spacetime

different gravitational attraction one can distinguish different positions and directions. If however, one restricts physical evolutions to such short times and small regions that the inconstancy of gravity does not make itself felt at the given precision of measurement, then the effects of gravity become imperceptible in a freely falling system of reference. In sufficiently small regions the curvature of spacetime is insensible and spacetime has the geometric properties of a flat space (Fig. 1.2).

We cannot shield gravity but want to avoid the related complications of a curved spacetime. Therefore we restrict our considerations to short times and distances such that gravitational effects are immeasurably small or we account mentally for the known gravitational effects and subtract them from the observed behavior of the physical systems.

If then, one has shielded all external influences and subtracted gravitational effects, then one cannot measure the time and the position of an event without reference to other events—just as little as at sea one cannot measure latitude without reference to the sun and time in Greenwich or without GPS. Physical evolutions are the same everywhere and at any time: the spacetime is homogeneous. Similarly physical evolutions are the same in all directions: the spacetime is isotropic.

Rotational Motion

Rotational motion, the temporal change of directions, can be measured—different from uniform straight motion—without reference to other bodies such as the distant stars. If one rotates and emits light into some direction, then the reflected light is seen to return from a different direction [32]. In a rotating cinema one projects into one direction and observes from a different one. Only for nonrotating observers does reflected light return from the direction into which it had been emitted. In rotating reference systems, light to and fro does not follow the same path. This property is used in interferometers, which measure rotation with a precision of 10^{-8} degrees per second [8].

The situation where some object orbits around the observer is different from his own rotation. In both situations the light rays from the object come from directions which change in the course of time, but if one does not rotate one sees each single light ray reflected from the object return from the direction into which it was sent.

Remarkably, these rotation-free reference frames defined by the local property of reflected light to return from the direction of emission coincide within high experimental precision with the systems in which the light from the distant stars comes from directions that, apart from parallax, aberration and the motion of the star, do not change in time. This is by no means self-evident and, measured with today's highest precision [13], does not hold if one orbits the rotating earth.

Lightcone

Long before Einstein's theory of relativity the principle of relativity was known, that one cannot distinguish by any effect of Newtonian mechanics whether an unaccelerated observer moves or rests.

However, one was convinced that this principle of relativity is valid only approximately because it was known since 1676 from Olaf Rømer's observation and interpretation of the orbital periods of the four large moons of Jupiter, Jo, Europa, Ganymed and Kallisto, that c, the speed of light in the vacuum, is finite. Therefore light was assumed to single out an absolute rest system, in which light propagated equally fast in all directions and in which its medium, the ether, rested. For an observer, moving with a velocity v with respect to the ether, light should propagate in different directions with different velocities ranging from $c - v$ to $c + v$.

This conclusion is obvious and *wrong*: no experiment has ever measured that a moving source of light emits light which in the vacuum propagates with different velocities in different directions. Moreover, never has the motion of the observer made him to register in the vacuum different velocities of light in different directions. This is the result of the seminal experiment of Albert Michelson.[1] The most astonishing property of the ether is that never one found a trace of it. Ether has all the properties of the vacuum, it is the vacuum. Light propagates in the vacuum with a velocity which does not allow to distinguish a stationary observer from a uniformly moving observer.

Principle of Relativity: *The speed of light in the vacuum does not depend on the motion of the source. No physical observation allows to distinguish an observer at rest from an uniformly moving observer.*

In the vacuum there is no faster or slower light. Light does not outrun light [7].

For example, in 1987, one observed a supernova in the Large Magellanic Cloud, SN 1987a, which exploded 160 000 years ago and where the luminous plasma was

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[1] Experimental findings are discussed in detail in [34].

emitted in all directions with a velocity of 25 000 km/s. If this velocity v had added to the velocity of light to $c' = c + v$, then the light from the plasma which moved towards us would have arrived 12 000 years earlier than the light from the plasma which moved transversal to the line of sight.

Nobody had observed the star when the first light of the explosion arrived here, but in the explosion also neutrinos were emitted whose time of arrival was recorded. As one later found, they had triggered the counters one hour before one looked into the direction of the star and saw the explosion. At that time it was completely visible. Thus, there could have been runtime differences of at most one hour for the different lightrays.

A year has roughly $365 \cdot 24$ hours. With a runtime of 160 000 years the velocities of the light rays therefore were equal up to $1/(160\,000 \cdot 365 \cdot 24) \approx 0{,}7 \cdot 10^{-9}$, i.e. in the first nine decimals.

By the way, this observation implies also [31] that the neutrinos had moved with the speed of light within this precision, and that their mass is less than 10 eV/c^2.

That the speed of light in the vacuum is independent of the speed of the source agrees with the conclusions which one can draw from Maxwell's equations (5.3, 5.4) for the electromagnetic fields. From these equations we deduce (page 97) that a charge which is at the position $\mathbf{x}'$ at the time t' influences the electric and magnetic fields at the position $\mathbf{x}$ at the time t,

$$c\,(t - t') = |\mathbf{x} - \mathbf{x}'|\ , \tag{1.1}$$

which is later than t' by the runtime of light $|\mathbf{x} - \mathbf{x}'|/c$. The time t does not depend on the velocity of the charge, which causes the field.

The events $(t, \mathbf{x})$ constitute the forward lightcone of the event $(t', \mathbf{x}')$; electromagnetic causes produce effects with the speed of light in the vacuum.

The independence of the propagation of light from the velocity of the source does not imply that other properties, such as the color of the light, the direction of the light rays and the intensity of the radiation, do not depend thereon. The direction of the incoming light rays, the color of the light and the number of photons per time and unit area, the luminosity, do depend on the velocity of the source and the velocity of the receiver. The intensity of electromagnetic radiation depends on the acceleration of the charges that emit the radiation.

The independence of the propagation of light from the velocity of the source is illustrated in the spacetime Fig. 1.3. A stationary observer traverses the straight worldline $\mathcal{O}_0$, his position x remains unchanged at all times t; a second observer traverses the worldline $\mathcal{O}_v$ and moves uniformly into x-direction. If both observers are at some time at the same position and emit a flash of light, then the light propagates from this event E equally fast in all directions[2] and independently of whether the light source moves.

[2] Our diagrams show only one spatial dimension. Therefore there is only the direction forwards or backwards.