
Geometric Partial Differential Equations proceedings

edited by

Antonin Chambolle, Matteo Novaga
and Enrico Valdinoci



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Preface

This book is the outcome of a conference held in September 2012 at the Centro De Giorgi of the Scuola Normale in Pisa. The aim of the conference was to discuss recent results on nonlinear analysis, and more specifically on geometric partial differential equations.

These equations arise in several models from physics, biology, material science, image processing and applied mathematics in general, and attracted a lot of attention in recent years. The typical question is studying a set or a function with particular properties (isoperimetric problems, shape optimization, free boundary problems, minimal surfaces), or the way a set or a function evolves (front propagation, mean curvature flow, total variation flow, travelling waves, Ginzburg-Landau problems).

A characteristic feature of these problems is the ubiquitous onset of singularities, and a detailed analysis of this phenomenon is often a very difficult task. As classical solutions are generally smooth, the presence of singularities motivates the definition of weak solutions, among which we mention viscosity solutions, varifold solutions and currents. The analysis of these objects requires the development of advanced tools from different branches of mathematics, such as calculus of variations, geometric measure theory, partial differential equations and differential geometry.

In this book, particular attention is paid to the study of special self-similar solutions, such as solitons, homothetic self-shrinkers and travelling waves, which are crucial for understanding both the nature of singularities and the asymptotic behaviour of solutions. Qualitative properties of solutions, such as convexity or monotonicity, are also considered.

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On the structure of phase transition maps for three or more coexisting phases

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Abstract. We consider system (1.1) below. After stating certain general facts which do not depend on the structure of W , we focus on the phase transition case. For symmetric W 's the main issues have been resolved, and we summarize them here. Next we recall the De Giorgi conjecture in the scalar case, and we point to a Bernstein type theorem that appears appropriate for (1.1). Finally we state a result on the hierarchical structure of the equivariant solutions.

1 Introduction

This paper is partly based on a lecture delivered by the author at the ERC workshop “Geometric Partial Differential Equations” held in Pisa in September 2012. What is presented in the following is an expanded version of that lecture.

Specifically, we consider the system

$$\Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad (1.1)$$

with $W \in C^2(\mathbb{R}^m; \mathbb{R}_+)$, $W \geq 0$, where $W_u := (\partial W / \partial u_1, \dots, \partial W / \partial u_n)^\top$, and occasionally with additional hypotheses introduced later on. We refer the reader to Part I in [9] for general information and motivation for system (1.1).

The paper is organized as follows. In Section 2 we present the basics of the problem for general potentials, in Section 3 we study symmetric potentials for the phase transition model and establish the existence of equivariant connection maps, in Sections 4 and 5 we present and prove a related Bernstein-type theorem, and in Section 6 we discuss the hierarchical structure of the equivariant connection maps. Sections 2, 3, and 6 are restricted to statements of results with explanations but without proofs, referring to published papers or preprints for the details. In contrast, in Sections 4 and 5 we give the background and detailed proofs.

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2 The basics for general potentials

We recall some known facts for (1.1). The system is the Euler–Lagrange equation for the *free energy functional*

$$J(u; \mathbb{R}^n) := \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx,$$

where $\nabla u = (\partial u_i / \partial u_j)$, for $i = 1, \dots, m$, $j = 1, \dots, n$, and $|\cdot|$ is the Euclidean norm of the matrix.

One important difference of (1.1) with its scalar counterpart (for $m = 1$) is that in that case the structure of bounded entire solutions does not depend very much on W . In contrast, for the system there two distinguished examples with very distinct behavior: The class of *phase transition potentials*, that is, W 's with a finite number of global minima (wells) $a_1, \dots, a_N$, with $W(a_i) = 0$, and the class of *Ginzburg–Landau potentials*, for example, the potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ (disconnected versus connected zero sets). In the phase transition case and under suitable rescaling the free energy concentrates on minimal hypersurfaces or Plateau complexes (see [14]), while in the Ginzburg–Landau case it concentrates on higher-codimension objects known as vortices, and otherwise the solution converges to a harmonic map (see [18]).

Equation (1.1) can be written as a divergence-free condition, that is,

$$\operatorname{div} T = (\nabla u)^\top (\Delta u - W_u(u)) = 0,$$

for the stress-energy tensor

$$T_{ij}(u, \nabla u) := u_{,i} \cdot u_{,j} - \delta_{ij} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right).^1$$

In this context it was introduced in [3], but as it turns out it is a particularization of a general formalism well-known to the physicists [40].

¹ The sharp-interface limit $T_\varepsilon \rightarrow T_0 = \sigma(\nabla d \otimes \nabla d - \operatorname{Id})$ is the orthogonal projection to the tangent space of the interface S separating the two phases, where $d = \operatorname{dist}(x, S)$ and σ is the associated interface energy.

The divergence-free formulation has certain important consequences. For example, one can derive the monotonicity formula

$$\frac{d}{dR} \left(\frac{1}{R^{n-2}} \int_{|x-x_0|<R} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \right) \geq 0,$$

from which Liouville-type theorems follow (see [3]). For instance,

$$\begin{cases} J(u; B_R) = o(R^{n-2}) \text{ as } R \rightarrow +\infty, \text{ for } n \geq 3, & \text{implies } u \equiv \text{constant}, \\ J(u; B_R) = o(\log R) \text{ as } R \rightarrow +\infty, \text{ for } n = 2, & \text{implies } u \equiv \text{constant}. \end{cases}$$

In particular,

$$\int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx < +\infty \quad \text{implies} \quad u \equiv \text{constant}, \quad (2.1)$$

for $n \geq 2$. Note that (2.1) is a source of difficulty for constructing solutions to (1.1) via the direct method. It was Farina [27] who first derived the monotonicity formula above and its implication (2.1) in the context of the Ginzburg–Landau system and Modica [41] who derived (2.1) for $m = 1$.

In the scalar ODE case ($n = 1, m = 1$), for solutions of (1.1) with limits at infinity, one has the elementary *equipartition relation*

$$\frac{1}{2} |u_x|^2 = W(u).$$

In the scalar PDE case ($n \geq 2, m = 1$), Modica [41] established the estimate

$$\frac{1}{2} |\nabla u|^2 \leq W(u), \quad (2.2)$$

(see also [20]). The analog of estimate (2.2) is false for systems in general. All the known counterexamples (see [27, pages 389–390]) involve Ginzburg–Landau potentials. One implication of (2.2) would be the stronger monotonicity formula

$$\frac{d}{dR} \left(\frac{1}{R^{n-1}} \int_{|x-x_0|<R} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \right) \geq 0,$$

already known for the scalar case (see [42]).

Another implication of the divergence-free formulation is a Pohozaev-type identity (see [7])

$$\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 dx + n \int_{\Omega} W(u) dx + \frac{1}{2} \int_{\partial\Omega} (x - x_0) \cdot \nu |\nabla u|^2 dS = 0,$$

where ν is the outward normal and $x_0 \in \Omega$ arbitrary, for solutions of the system

$$\begin{cases} \Delta u - W_u(u) = 0, & \text{in } \Omega \subset \mathbb{R}^n, \\ u = a, & \text{on } \partial\Omega, \text{ with } W(a) = 0. \end{cases}$$

Gui [36] has developed certain identities for the system, which he calls ‘Hamiltonian’, and points out their relationship with the classical Pohozaev identity (see, for example, [26]). His identities can also be derived via the stress-energy tensor. Here is a sample: Let $n = 2, m = 2$, and let u be a solution to (1.1) satisfying the estimate

$$|u(x_1, x_2) - a_{\pm}| \leq C e^{-c|x_1|}, \text{ for all } x_2 \in [M, N],$$

with $-\infty \leq M, N \leq +\infty$ and $W(a_{\pm}) = 0$. Then,

$$\int_{\mathbb{R}} \left(\frac{1}{2} (|u_{x_1}(x_1, x_2)|^2 - |u_{x_2}(x_1, x_2)|^2) - W(u(x_1, x_2)) \right) dx_1 = \text{constant},$$

for all $x_2 \in (M, N)$.

Having asymptotic information on the solution along certain hyperplanes as $|x| \rightarrow +\infty$ and also convergence to a minimum of W away from them (see Section 4), one can measure the flux of the stress-energy tensor over large spheres in order to derive balance conditions relating the angles between the hyperplanes, thus deriving rigidity-type results. For the phase transition case and for a triple-well potential, Gui [36] has derived such a result in the planar case $n = 2, m = 2$, thus relating the angles of a triple junction to the surface energies. This was extended to the three-dimensional case $n = 3, m = 3$, in [5]. Related also is the work of Kowalczyk, Liu, and Pacard [39].

3 Symmetric phase transition potentials

Existence of equivariant connection maps

In this section we restrict ourselves to the phase transition case for potentials that respect the symmetries of a finite reflection group G acting on $\mathbb{R}^n$ (see [35]) and we look for equivariant solutions

$$u(gx) = gu(x), \text{ for all } x \in \mathbb{R}^n \text{ and } g \in G.$$

The first results in this direction are due to Bronsard, Gui, and Schatzman [19] for $n = 2, m = 2$, and G the group of reflections of the equilateral triangle. Later, the work was extended by Gui and Schatzman [37] to $n = 3, m = 3$, and G the group of symmetries of the regular tetrahedron. These two special groups are particularly important as they

are related to triple junctions on the plane and to quadruple junctions in three-dimensional space, which are minimal objects (cones) for the related sharp-interface problem.

In work with Fusco [10] we considered the general case of a reflection group and looked for an abstract result. Consider the following very general hypotheses.

Hypothesis 1 (N nondegenerate global minima). *The potential W is of class C^2 and satisfies $W(a_i) = 0$, for $i = 1, \dots, N$, and $W > 0$ on $\mathbb{R}^m \setminus \{a_1, \dots, a_N\}$. Furthermore, there holds $v^\top \partial^2 W(u) v \geq 2c^2 |v|^2$, for $v \in \mathbb{R}^m$ and $i = 1, \dots, N$.*

Hypothesis 2 (Symmetry). *The potential W is invariant under a finite reflection group G acting on $\mathbb{R}^m$ (Coxeter group), that is,*

$$W(gu) = W(u), \text{ for all } g \in G \text{ and } u \in \mathbb{R}^m.$$

Moreover, we assume that there exists $M > 0$ such that $W(su) \geq W(u)$, for $s \geq 1$ and $|u| = M$.

Hypothesis 3 (Location and number of global minima). *Let $F \subset \mathbb{R}^m$ be a fundamental region of G . We assume that the closure $\overline{F}$ contains a single global minimum of W , say a_1 , and let G_{a_1} be the subgroup of G that leaves a_1 fixed.*

We set

$$D = \text{Int}\{\cup g\overline{F} \mid g \in G_{a_1}\},$$

and notice that by the invariance of W it follows that the number of minima of W is

$$N = \frac{|G|}{|G_{a_1}|},$$

where here $|\cdot|$ is the order of the group.

We recall from [10] several examples. For $G = \mathcal{H}_2^3$, the group of symmetries of the equilateral triangle on the plane, we can take as F the $\frac{\pi}{3}$ sector. If $a_1 \in F$, then $N = 6$, while if a_1 is on the walls, then $N = 3$. In higher dimensions we have more options since we can place a_1 in the interior of $\overline{F}$, in the interior of a face, on an edge, and so on. For example, if $G = \mathcal{W}^*$, the group of symmetries of the cube in three-dimensional space, then $|G| = 48$. If the cube is situated with its center at the origin and its vertices at the eight points $(\pm 1, \pm 1, \pm 1)$, then we can take as F the simplex generated by $s_1 = e_1 + e_2 + e_3$, $s_2 = e_2 + e_3$, and $s_3 = e_3$, where the e_i 's are the standard basis vectors. We have then the following options:

- (i) At the origin, $N = 1$.
- (ii) On the edge s_3 , $N = 6$.

- (iii) On the edge s_1 , $N = 8$.
- (iv) On the edge s_2 , $N = 12$.
- (v) In the interior of a face, $N = 24$.
- (vi) In the interior of the fundamental region, $N = 48$.

We have the following theorem.

Theorem 3.1 ([10, 4, 33]). *Under Hypotheses 1–3, there exists a classical entire equivariant solution $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to system (1.1) such that*

- (i) $|u(x) - a_1| \leq K e^{-k \operatorname{dist}(x, \partial D)}$, for $x \in D$ and for positive constants k, K ,
- (ii) $u(\overline{F}) \subset \overline{F}$ and $u(D) \subset D$ (positivity).²

As a consequence of (i), the solution u connects the $N = |G|/|G_{a_1}|$ global minima of W in the sense that

$$\lim_{\lambda \rightarrow +\infty} u(\lambda g \eta) = g a_1, \text{ for all } g \in G,$$

uniformly for η in compact subsets of $D \cap \mathbb{S}^{n-1}$.

Remark 3.2. We need a clarification concerning the dimensions n, m and the group G that is acting by Hypothesis 2 on the target but also on the domain since the solutions are equivariant. If $n \geq m$, then the group G can be embedded in the domain space via a natural homomorphism. For example, consider $n = 3, m = 2$, and G the group of symmetries of the equilateral triangle. On the other hand, if $n < m$, the existence of such a homomorphism is more problematic and in general there is no such embedding. For example, consider $n = 2, m = 3$, and take as G the group associated to the tetrahedron. For relevant information we refer to Bates, Fusco, and Smyrnelis [15]. Our notation $u(\overline{F})$ and $u(\lambda g \eta)$ tacitly assumes the homomorphism in the case $n \neq m$.

The theorem above was proved in Alikakos and Fusco [10] under an additional hypothesis. Subsequently, the author gave a simplified proof in [4] and, finally, in Fusco [33] the extra hypothesis was removed and the theorem was proved under the hypotheses above.

As it was mentioned in (2.1), there holds

$$J(u; \mathbb{R}^n) = +\infty, \text{ where } J(u, \Omega) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx,$$

² Smyrnelis has established that $u(F) \subset F$ for certain groups (personal communication).

for the solution constructed above. However, the solutions are constructed variationally and possess the following minimization property (see [11]), which defines the notion of a *local minimizer* (cf. [2]), that is,

$$J(u; \Omega) = \min J(v; \Omega), \text{ such that } v = u \text{ on } \partial\Omega, \quad (3.1)$$

over all domains Ω that are bounded, smooth, and open, but not necessarily symmetric, and over all equivariant positive maps v (cf. (ii) in the statement of Theorem 3.1) in $W^{1,2}(\Omega; \mathbb{R}^m)$.

Remarks 3.3. The constructed solution u is a minimizer in the equivariant positive class. For the equilateral triangle group on the plane and the regular tetrahedron group in three-dimensional space, one would expect that the solution constructed is a minimizer in the class of all $W^{1,2}(\Omega; \mathbb{R}^n)$ -maps.

It appears that the positivity of u is not an implication of the minimizing property (3.1) for arbitrary G , without extra hypotheses on W .

Looking for equivariant solutions is of course a convenience. However, at this point all the existence results for system (1.1) known to the author involve hypotheses of symmetry (except for $n = 1$).

4 A related Bernstein-type theorem – Background

We recall De Giorgi's conjecture [23] for the scalar equation, for $n \geq 2$, $m = 1$.

Conjecture 4.1 (De Giorgi). For the equation

$$\Delta u - W'(u) = 0,$$

with $W(u) = \frac{1}{4}(u^2 - 1)^2$, under the hypotheses that $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is in $C^2(\mathbb{R}^n; [-1, 1])$ with $\partial u / \partial x_n > 0$, is it true that the level sets of u are hyperplanes, at least for $n \leq 8$?

A more restricted version of the conjecture involves the additional hypothesis $\lim_{x_n \rightarrow \pm\infty} u(x) = \pm 1$.

This conjecture was established by Ghoussoub and Gui [34] for $n = 2$, Ambrosio and Cabré [13] for $n = 3$, and Savin [44] in the restricted form for $4 \leq n \leq 8$. Finally, it was disproved for $n \geq 9$ by del Pino, Kowalczyk, and Wei [24, 25]. We refer to the survey paper by Farina and Valdinoci [30], where in addition several extensions to a variety of related equations are given.

Some of the ingredients behind the formulation of this conjecture are

- (i) the Bernstein theorem for graphs,
- (ii) the relationship between monotonicity and stability,

(iii) the solution of the ODE (the *heteroclinic connection*)

$$\frac{d^2 U}{d\eta^2} - W'(U) = 0, \text{ with } \lim_{\eta \rightarrow \pm\infty} U(\eta) = \pm 1$$

(unique up to translations),

(iv) the phase transition problem for two phases.

The conclusion in the conjecture is equivalent to showing that

$$u(x) = U\left(\frac{a \cdot x - c}{\sqrt{2}}\right),$$

for some $a \in \mathbb{R}^n$, with $|a| = 1$, and $c \in \mathbb{R}$, that is, $u(x) = U(Px)$, where P is the orthogonal projection to the normal direction of the level sets.

In formulating the analog of the conjecture for systems one should keep in mind that

- (i) Tangent planes are special cases of tangent cones. Moreover, minimizing tangent cones have cylindrical structure, that is,

$$C = V \times \tilde{C}, \text{ with } \tilde{C} \text{ minimizing in } V^\perp,$$

and the cone C is translation invariant ‘along V ’ ($V = \{0\}$ is an option). Also, the Liouville object in this context is the cone at infinity.

The Bernstein-type theorem therefore should involve a cone.

- (ii) Monotonicity is not related in general to stability for systems.
- (iii) The solution $u : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ *a posteriori* should be of the form

$$u(x) = \hat{u}(Px),$$

where P is an orthonormal projection on an $(n - k)$ -dimensional plane, $\hat{u} : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k}$ is a *connection* map, equivariant as in Theorem 3.1 in Section 3.

- (iv) For three or more phases the order parameter should be a vector since otherwise there is no connection between the extreme phases. For example, for coexistence of three phases we need at least a two-dimensional order parameter, thus a partitioning of $\mathbb{R}^2$ in three parts.
- (v) For the analog of the restricted conjecture we refer to Section 5 in the present paper.

Our purpose in this section is to present a sample of such Bernstein-type theorems for the simplest nontrivial case, the triple junction in $\mathbb{R}^3$, that corresponds to one of the two singular minimizing cones in $\mathbb{R}^3$ (see [50]). The formulation of such theorems in terms of cones goes back to Fleming

[32] and is subsequently developed in Morgan [43]. However, here we also want to emphasize partitions as the natural setup. For this reason we present in detail White's approach [51] because it improves Almgren's [12] and because of its simplicity, and also for making our treatment as self-contained as possible. Our presentation here is also based on Chan's thesis [21], written under White's supervision.

Finally, we mention the paper of Fazly and Ghoussoub [31], which extends the methods of the scalar equation to systems, as far as this can possibly be done, by assuming certain monotonicities on the components of the solution which amount to

$$\frac{\partial^2 W(u)}{\partial u_i \partial u_j} \leq 0, \text{ for } i \neq j.$$

For a special system of two equations, we mention the papers of Berestycki, Lin, Wei, and Zhao [16], Berestycki, Terracini, Wang, and Wei [17], Farina [28], and Farina and Soave [29].

We recall here some basic background on partitions and geometric measure theory (see [45, 51, 52]).

4.1 Minimizing partitions

Consider an open set $U \subset \mathbb{R}^n$ occupied by N immiscible fluids, or phases. Associated to each pair of phases i and j there is a surface energy density e_{ij} , with $e_{ij} > 0$ for $i \neq j$, and $e_{ij} = e_{ji}$, with $e_{ii} = 0$. Hence, if A_i denotes the subset of U occupied by phase i , then U is the disjoint union

$$U = A_1 \cup A_2 \cup A_2 \cup \cdots \cup A_N,$$

and the energy of the partition $A = \{A_i\}_{i=1}^N$ is

$$E(A) = \sum_{0 < i < j \leq N} e_{ij} \mathbb{M}(\partial A_i \cap \partial A_j),$$

where $\mathbb{M}$ (for *mass*) stands for the measure of the interface. For $n = 3$ it will simply be the area of $\partial A_i \cap \partial A_j$.

If U is unbounded, for example $U = \mathbb{R}^n$ (we say then that A is *complete*), the quantity above in general will be infinity. Thus, for each W open, with $W \Subset U$, we consider the energy

$$E(A; W) = \sum_{0 < i < j \leq N} e_{ij} \mathbb{M}(I_{ij} \cap W), \text{ where } I_{ij} := \partial A_i \cap \partial A_j.$$

Definition 4.2. The partition A is a *minimizing* N -partition if given any $W \subseteq U$ and any N -partition A' of U with

$$\bigcup_{i=1}^N (A_i \triangle A'_i) \subseteq W, \quad (4.1)$$

we have

$$E(A; W) \leq E(A'; W).$$

The symmetric difference $A_i \triangle A'_i$ of the sets A_i and A'_i is defined as their union minus their intersection, that is, $A_i \triangle A'_i = (A_i \cup A'_i) \setminus (A_i \cap A'_i)$.

4.2 Flat chains with coefficients in a group

Let G be an abelian group with norm $|\cdot|$, such that $|g| \geq 0$, with $|g| = 0$ if and only if $g = 0$, for all $g \in G$, and

$$|g + h| \leq |g| + |h|, \text{ for all } g, h \in G.$$

Then, $(G, |\cdot|)$ is a metric space and we will assume that it is complete and separable. In our case G will be a finite group.

Fix $\mathbb{R}^n$ and a compact convex set $\mathbb{K}$ in $\mathbb{R}^n$. For each integer $k \geq 0$ consider the abelian group of all formal finite sums of the form $\sum g_i P_i$, where $g_i \in G$ and where P_i is a k -dimensional oriented compact convex polyhedron in $\mathbb{K}$. We form the quotient group obtained by identifying gP with $-g\tilde{P}$, whenever P and $\tilde{P}$ coincide but have opposite orientations. Also, identify gP and $gP_1 + gP_2$, whenever P can be subdivided into P_1 and P_2 .

The resulting abelian group $\mathcal{P}_k(\mathbb{K}; G)$ is called the group of *polyhedral k -chains* on $\mathbb{K}$ with coefficients in G . Define the *boundary homomorphism* $\partial : \mathcal{P}_k \rightarrow \mathcal{P}_{k-1}$ by

$$\partial \left(\sum g_i P_i \right) := \sum g_i \partial P_i.$$

Note that any polyhedral k -chain T can be written as a linear combination $\sum_i g_i [P_i]$ of nonoverlapping polyhedra, that is, polyhedra with disjoint interiors. Then, the *flat norm* of the chain is defined to be

$$W(T) = \inf_Q \{ \mathbb{M}(T - \partial Q) + \mathbb{M}(Q) \},$$

where the infimum is over all polyhedral $(k+1)$ -chains Q .

The flat norm makes $\mathcal{P}_k(\mathbb{K}; G)$ into a metric space. The completion of this metric space is denoted by $\mathcal{F}_k(\mathbb{K}; G)$ and its elements are called *flat k -chains* in $\mathbb{K}$ with coefficients in G . By uniform continuity, functionals

such as the flat norm and operations such as addition and boundary extend in a unique way from polyhedral chains to flat chains. The mass norm in $\mathcal{P}_k(\mathbb{K}; G)$ extends to a linear semicontinuous functional in $\mathcal{F}_k(\mathbb{K}; G)$.

Suppose that every bounded closed subset of G is compact. A fundamental compactness theorem for flat chains asserts that, given any sequence $T_i \in \mathcal{F}_k(\mathbb{K}; G)$ with $\mathbb{M}(T_i)$ and $\mathbb{M}(\partial T_i)$ uniformly bounded, there is a W -convergent subsequence. More generally, one can define the flat chains in $\mathbb{R}^n$ with compact support, $\mathcal{F}_k(\mathbb{R}^n; G)$, meaning that each element vanishes outside a certain compact convex set. Then, the compactness theorem holds for a sequence $T_i \in \mathcal{F}_k(\mathbb{R}^n; G)$, with $\text{supp } T_i \subset \mathbb{K}$, for $\mathbb{K}$ independent of i . The symbol ‘ $\rightarrow$ ’ denotes convergence in the flat norm.

4.3 Flat chains of top dimension

[See [51].] Polyhedral n -chains in $\mathbb{R}^n$ with compact support can be identified with the set of piecewise-constant functions

$$g : \mathbb{R}^n \rightarrow G,$$

that vanish outside a compact convex set $\mathbb{K}$. Here, two functions that differ only on a set of measure zero are regarded as the same. ‘Piecewise constant’ means locally constant except along a finite collection of hyperplanes. The identification is as follows. Any such $T \in \mathcal{F}_k(\mathbb{R}^n; G)$ can be written as

$$T = \sum g_i [P_i],$$

where the P_i ’s are nonoverlapping and inherit their orientations from $\mathbb{R}^n$. We can associate to T the function

$$g : \mathbb{R}^n \rightarrow G, \text{ with } g(x) = \begin{cases} g_i, & \text{if } x \text{ is in the interior of } P_i, \\ 0, & \text{if } x \text{ is not in the interior of } P_i. \end{cases}$$

Note that the mass norm of T is equal to the L^1 norm of $g(\cdot)$. Also, since there are no nonzero $(n+1)$ -chains in $\mathbb{R}^n$, we see from the definition of W that

$$W(T) = \mathbb{M}(T) = \int_{\mathbb{R}^n} |g(x)| \, dx.$$

Consequently, the W -completion of the polyhedral chains (that is, the flat n -chains) is isomorphic to the L^1 -completion of the piecewise-constant functions.

Denoting T by $[\mathbb{K}]_{Lg}$, the isomorphism is

$$L^1(\mathbb{K}; G) \ni g \rightarrow [\mathbb{K}]_{Lg} \in \mathcal{F}_n(\mathbb{K}; G),$$

with

$$\mathbb{M}([\mathbb{K}]_{Lg}) = W([\mathbb{K}]_{Lg}) = \int_{\mathbb{R}^n} |g| \, dx.$$

Thus, the flat n -chains T on $\mathbb{R}^n$ with compact support can be identified with the $L^1_{\text{loc}}(\mathbb{R}^n; G)$ functions. The flat chains with $\mathbb{M}(\partial T) < +\infty$ correspond to the sets with finite perimeter (Caccioppoli sets). The BV norm of the function g above gives the perimeter, that is,

$$\|g\|_{BV} = \mathbb{M}(\partial T).$$

The compactness for flat chains with

$$\mathbb{M}(T_n) + \mathbb{M}(\partial T_n) < C$$

is equivalent *in this setup* to the compactness of the embedding

$$BV(\Omega) \Subset L^1(\Omega), \text{ for } \Omega \text{ bounded.}$$

The lower semicontinuity of $\mathbb{M}(\partial T)$ with respect to the W -norm is equivalent to the lower semicontinuity of the BV norm with respect to L^1 .

4.4 The group of surface tension coefficients (see [51])

The purpose next is the introduction of an appropriate group G so that for the flat chain $T = \sum g_i P_i$, where $P_i = A_i$, with $A = \{A_i\}$ a partition of U , there holds

$$\mathbb{M}(\partial T \llcorner W) = E(A; W). \quad (4.2)$$

First, assume that

$$e_{ik} \leq e_{ij} + e_{jk}, \text{ for all } i, j, k. \quad (4.3)$$

Let G be the free $\mathbb{Z}_2$ -module with N generators $f_1, \dots, f_N$ (one for each phase). White [51] defines a norm in this group such that

$$|f_i - f_j| = e_{ij},$$

and the $\mathbb{Z}_2$ -module identifies

$$f_{i_1} - f_{j_1} = f_{i_1} + f_{j_1}.$$

Utilizing this, it is easy to see in calculating ∂T , and $\mathbb{M}(\partial T)$, that (4.2) holds. In this setup, given a partition of U into N measurable sets $A_1, \dots, A_N$, and $\mathbb{K}$ as above, we associate the flat n -chain

$$T = \mathbb{K}_{Lg},$$

where

$$g(x) = \begin{cases} f_i, & \text{for } x \in A_i \cap \mathbb{K} \\ 0, & \text{for } x \notin A_i \cap \mathbb{K}. \end{cases}$$

Note that if the A_i 's have piecewise-smooth boundaries, then (4.2) holds. More generally, equation (4.2) holds whenever the A_i 's are Caccioppoli sets, that is, whenever the flat chains have finite mass.

Conversely, given any flat n -chain T , we can represent T as

$$T = \mathbb{K}_{Lg},$$

where $g \in L^1(U \cap \mathbb{K}; G)$. In this article we take $U = \mathbb{R}^n$. We note that in [51] it is shown that the inequalities (4.3) are no real restriction, in the sense that if they are violated, then one can define new coefficients e_{ij}^* out of the old, so that the infimum of E coincides with the infimum of E^* (defined by replacing e_{ij} with e_{ij}^*). Also, it is noted that (4.3) is necessary for E to be lower semicontinuous with respect to the flat norm. Here, we refer also to Section 4.1 in [1].

4.5 Basics on minimizing chains

We recall some standard facts on minimizing chains and later we point out the relationship with minimizing partitions.

Cones If $x_0 \in \mathbb{R}^n$, where S is a k -dimensional flat chain in $\mathbb{R}^n$, then *the cone over S with vertex at x_0* is the flat chain

$$x_0 S = \text{Cone}(S) = h(I \times S), \quad (4.4)$$

where $h(t, x) = (1 - t)x_0 + tx$, for $0 \leq t \leq 1$, and $x \in S$. We have

$$S = \partial(x_0 S) + x_0 \partial S,$$

and if $S \subset B_r(x_0)$, where $B_r(x_0)$ is the ball with radius r and center at x_0 , then

$$\mathbb{M}(x_0 S) \leq \frac{r}{k+1} \mathbb{M}(S).$$

C_x is a cone with vertex at x if, by definition, it is invariant as a set under the homothetic map

$$y \rightarrow x + t(y - x), \text{ for all } t > 0 \text{ and } y \in C_x.$$

If S is a k -flat chain in $\mathbb{R}^n$, then S is *mass minimizing* if $\mathbb{M}(S) \leq \mathbb{M}(S')$, for all S' with $\partial S' = \partial S$.

If Γ is a *cycle*, that is, $\partial\Gamma = 0$, then

$$L(\Gamma) := \inf\{\mathbb{M}(X) \mid \partial X = \Gamma\}.$$

Consequently, if S is mass minimizing, then $\mathbb{M}(S) = L(\partial S)$.

The flat chain S is *minimizing* if by definition

$$\mathbb{M}(S \llcorner B_r(x)) = L(\partial(S \llcorner B_r(x))),$$

for all $r > 0$ and center x such that $0 < r < \text{dist}(x, \text{supp } \partial S)$. Mass minimizing is minimizing. We allow the options $\partial S = 0$ and $\mathbb{M}(S) = \infty$.

The *monotonicity formula* holds for k -dimensional minimizing flat chains and states that

$$\Theta(S, x, r) := \frac{\mathbb{M}(S \llcorner B_r(x))}{\omega_k r^k} \quad (4.5)$$

is an increasing function of r , where ω_k is the volume of the k -dimensional unit ball. It follows that for minimizing flat chains S , the limit

$$\Theta(S, x) := \lim_{r \rightarrow 0} \Theta(S, x, r)$$

exists, and if

$$\Theta(S, x, r) < B, \text{ for } B \text{ independent of } x, r, \quad (4.6)$$

then the limit

$$\Theta(S) := \lim_{r \rightarrow +\infty} \Theta(S, x, r)$$

exists and is independent of x . We note that the condition $\Theta(S, x, r) = \text{constant}$ in $r > 0$, for x fixed, implies that S is a cone with vertex at x .

The tangent cone (blow-up) Let S be a minimizing flat chain, $x \notin \text{supp } \partial S$, and let $\{\mu_i\}$ be an increasing sequence of positive numbers, with $\mu_i \rightarrow +\infty$. Set

$$S_i = \mathcal{D}_{\mu_i}(S - x), \text{ with } \mathcal{D}_{\mu_i}(S - x) = \{\mu_i(y - x) \mid y \in S\}.$$

Then *along a subsequence* there holds $S_i \rightarrow C_x$ (by the compactness theorem), where C_x has the properties

- (i) $\partial C_x = 0$,
- (ii) C_x is a cone,
- (iii) $\Theta(C, 0, r) = \Theta(S, x)$, for all $r > 0$,
- (iv) C_x is minimizing.

The cone at infinity (blow-down) If instead in the arrangement above $\{\mu_i\}$ is a decreasing sequence, with $\mu_i \rightarrow 0$, and if (4.6) holds, then along a subsequence there holds $S_i \rightarrow C_\infty$ (by the compactness theorem), where C_∞ has the properties

- (i) $\partial C_\infty = 0$,
- (ii) C_∞ is a cone,
- (iii) $\Theta(C_\infty, 0, r) = \Theta(C_\infty, 0) =: \Theta(S)$,
- (iv) C_∞ is minimizing.

Note that C_x and C_∞ are not necessarily unique.

If N^{k-1} is a smooth $(k-1)$ -surface, with $k \leq n-1$ and $N^{k-1} \subset \mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$), then the *cone over* N^{k-1} is

$$C(N^{k-1}) = \left\{ x \in \mathbb{R}^n \mid \frac{x}{|x|} \in N^{k-1} \right\}.$$

If S is smooth, then the projection

$$\frac{1}{R}(S \cap \mathbb{S}_R^{n-1}),$$

of the set $S \cap \mathbb{S}_R^{n-1}$ on the unit sphere tends to C_∞ , as $R \rightarrow \infty$, provided that (4.6) holds, that is,

$$C_\infty = \lim_{R \rightarrow +\infty} C \left(\frac{1}{R} (S \cap \mathbb{S}_R^{n-1}) \right),$$

where the limit is in the flat norm, and exists along a sequence

$$R_1 < R_2 < \dots \rightarrow +\infty,$$

where C_∞ is the cone at infinity.

Relationship with partitions [See [12, 21, 46].] The concepts in Paragraph 4.5 have exact analogs for partitions defined as flat chains of top dimension in Paragraphs 4.3 and 4.4 above. Specifically,

- (i) the concept of the cone is unchanged,
- (ii) the mass minimizing flat chain S is replaced by the minimizing partition T (or A), via the definition in (4.1) above,
- (iii) the monotonicity formula holds for minimizing partitions,
- (iv) the notion of tangent cone and cone at infinity have exact analogs for minimizing partitions.

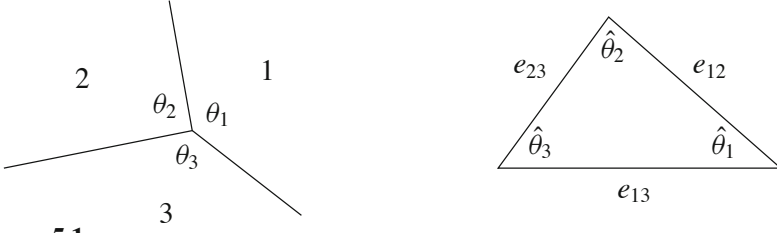


Figure 5.1.

5 A related Bernstein-type theorem – Statements and proofs

We are now ready to state a sample of a Bernstein-type theorem. We begin with $\mathbb{R}^2$.

Theorem 5.1 ($n = 2$). *Let A be a complete minimizing partition in $\mathbb{R}^2$ with $N = 3$ (three phases), with surface tension coefficients satisfying*

$$e_{ik} < e_{ij} + e_{jk}, \text{ for } j \neq i, k \text{ with } i, j, k \in \{1, 2, 3\}. \quad (5.1)$$

Then, ∂A is a triod.

In Figure 5.1 we show a triod with angles $\theta_1, \theta_2, \theta_3$, and the corresponding triangle with their supplementary angles $\hat{\theta}_i = \pi - \theta_i$. For these angles Young's law holds, that is,

$$\frac{\sin \hat{\theta}_1}{e_{23}} = \frac{\sin \hat{\theta}_2}{e_{13}} = \frac{\sin \hat{\theta}_3}{e_{12}}.$$

We recall that under the condition of the strict triangle inequality for the surface tension coefficients, White has established a general regularity result which applies in particular under (5.1) to A above. His result improves on Almgren's work [12]. Detailed proofs can be found in Chan's thesis [21] (Section 1.6 and pages 10–14). It follows that a priori A consists of triple junctions and line segments, always a finite number in any given open and bounded subset of $\mathbb{R}^2$.

We present the proof of Theorem 5.1 in three steps. The first two are two lemmas that we state next.

Lemma 5.2. *The only minimizing cones are the straight line and the triod.*

Lemma 5.3. *There holds $\mathbb{M}(\partial A \llcorner B_R) \leq CR$.*

Accepting for the time being the lemmas above, we can conclude with the proof of the theorem.