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Constructive Aspects of Functional Analysis

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

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CONSTRUCTIVE ASPECTS OF FUNCTIONAL ANALYSIS

Coordinatore : Prof. G. Geymonat

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

A. V. BALAKRISHNAN : I

A CONSTRUCTIVE APPROACH TO OPTIMAL CONTROL

Corso tenuto ad Erice dal 27 giugno al 7 luglio 1971

A CONSTRUCTIVE APPROACH TO OPTIMAL CONTROL

0. Introduction.

Except for linear problems, it is difficult, if not impossible, to obtain explicit solutions for optimal control problems. The closest we get to a general 'solution' is the Maximum Principle of Pontrjagin. But important as this result is, it only provides us with necessary conditions for a (any) postulated solution. Unfortunately, many control problems do not have an optimal solution. Consider for instance this trivial example:

$$\dot{x} = u ; \qquad x(0) = 0$$

Minimize:

$$\int_0^1 x^2(t)dt$$

subject to the constraint that the control $u(t)$ must be equal to $+1$ or -1 a.e. The minimal value is zero, but this is attained for $u(t) \equiv 0$ and of course this is impossible. On the other hand

$$u_n(t) = \frac{\text{Sin}\,\pi nt}{|\text{Sin}\,\pi nt|}$$

provides us with a sequence of admissible controls which approximate the infimum arbitrarily closely. The sequence $\{u_n(t)\}$ of course converge in the weak sense in $L_2[0,1]$ to zero, but unfortunately $u_n(t)^2$ converges to one, and of Course there is no optimal control.

In his recent book [1], L. C. Young has pointed out the fallacy in proving necessary conditions for a possibly non-existent solution. He cites a paradox of Perron that this leads to: consider the problem of finding the largest positive integer. If we assume there exists a solution, say N, then clearly $N \geq 1$; on the other hand, we must have that

$$N^2 \leq N$$

or,

$$N(N-1) \leq 0$$

or,

$$N-1 \leq 0$$

which combined with

$$N-1 \geq 0$$

shows that $N = 1$!

To resolve this difficulty, Young introduces the notion of a 'relaxed' or 'generalized control' and proves the existence of an optimal control in this class, and derives the maximum principle valid for such 'functions'. In the present work we shall go one step further and show how to actually construct - 'compute' - a sequence of approximating controls which converge to an optimal 'generalized control' and which then satisfies the maximum principle. The computational technique is of more than theoretical value; and in fact has proved to be practically useful as well.

Relaxed controls play an essential role in this approach. We begin with a simple exposition of the theory of relaxed controls [Young [1]], because it is of some independent interest as well.

1. Relaxed Controls

Let U be a compact set in Euclidean space E_m. Let H denote the L_2-space of functions $u(t)$, $0 < t < T < \infty$. Let $u_n(t)$ be any sequence of measurable functions such that

$$u_n(t) \in U \quad \text{a.e.}$$

Then we can find a subsequence (renumber it $u_n(\cdot)$) such that $u_n(\cdot)$ converges weakly to $u_0(\cdot)$ say in H. Let $p(\cdot)$ be any polynomial over E_m. Then

$$p[u_n(t)]$$

also contains a weakly convergent subsequence. What is the limit? Unfortunately it is not

$$p[u_0(t)]$$

as the example

$$u_n(t) = \frac{\text{Sin } nt}{|\text{Sin } nt|}$$

shows, taking $p(u) = u^2$. At the simplest level the 'generalized curves' [we shall continue to use the term generalized 'controls' because we shall need this notion only with the controls] may be regarded as providing a means to straighten out this situation.

Consider now the product space $\Omega = I \times U$ where I denotes the interval $[0, T]$. Then Ω is compact metric and let $C(\Omega)$ denote the Banach space of continuous functions over Ω with range in E_m. Let $f(t, u)$ denote such a function. Then observe that for any Lebesgue measurable function $u(t)$ such that

$$u(t) \in U \quad \text{a.e.}$$

we have that

$$\int_I f(t, u(t))dt$$

defines a continuous linear functional on $C(\Omega)$. We know that there must be a countably additive set function μ (of finite variation) defined on the Lebesgue subsets of Ω such that

$$\int_I f(t, u(t))dt = \int_\Omega f(t, u) \, d\mu(t, u)$$

and it is clear that μ is an atomic measure with a unit jump at $u(t)$ for each t. That is to say, on any product set of the form $\Delta \times B$

$$\mu(\Delta \times B) = \text{Lebesgue measure of the set } [t \mid u(t) \in B]$$

For any polynomial $p(\cdot)$, we note that

$$\int_U p(u)\, d\mu(t;u)$$

is Lebesgue measurable in t. A generalized control is simply a measure on (the Lebesgue subsets of) $I \times U$ such that

$$\int_U d\mu(t;u) = 1 \quad \text{a.e.}$$

and

$$\int_U p(u)\, d\mu(t;u)$$

is Lebesgue measurable in t. Alternately, for our purposes it is more natural to define it as a 'family' of probability measures ('control measures') $d\mu(t;u)$ over U such that

$$\int_U p(u)d\mu(t;u)$$

is Lebesgue measurable in t. Thus defined it is not difficult to show that

$$\int_\Omega f(t;u)\, d\mu(t;u)$$

defines a continuous linear functional on $C(\Omega)$. Moreover

$$\int_U f(t;u)\, d\mu(t;u)$$

is Lebesgue measurable in t.

Let now $u_n(t)$ be the sequence we began with, $u_n(t)$ converging weakly in H to $u_0(t)$. Let $f(t)$ be any $m \times m$ matrix function, continuous on I and $p(\cdot)$ be any polynomial with domain and range in E_m. Then we can write

$$\int_I f(t)\, p(u_n(t))dt$$

as

$$= \int_\Omega f(t)\, p(u)\, d\mu_n(t;u)$$

where $d\mu_n(t;u)$ is the corresponding sequence of measures. Now by the weak compactness of measures we know that (independent of $f(\cdot)$ and

$p(\cdot)$) we can find a subsequence (renumber it $d\mu_n(\cdot)$ again) which converges to a measure $d\,\mu_0(t;u)$:

$$\underset{n}{\text{limit}} \int_I f(t)\, p[u_n(t)]\, dt = \int_\Omega f(t)\, p(u)\, d\,\mu_0(t;u)$$

Working with a further subsequence, we know that

$$\underset{n}{\text{limit}} \int_I f(t)\, p[u_n(t)] dt = \int_I f(t)\, v(t)\, dt$$

where $v(t)$ is Lebesgue measurable and since $f(t)$ is arbitrary, it follows that

$$v(t) = \int_U p(u)\, d\mu_0(t;u)$$

Thus if we agree to define

$$p(u_0(t)) \doteq \bar{p}[u_0(t)] = \int_U p(u)\, d\,\mu_0(t;u)$$

where the bar indicates use of 'generalized control', then we do indeed have that if

$$u_n(t) \rightharpoonup u_0(t)$$

then

$$p(u_n(t)) \rightarrow p[u_0(t)]$$

Example

Let us illustrate this with a simple example for m = 1.

Let

$$u_n(t) = \frac{\operatorname{Sin} \pi nt}{|\operatorname{Sin} \pi nt|} \qquad 0 < t < 1$$

what is the limiting generalized function? Note that $d\,\mu_n(t;u)$ for each t has a jump at + 1 or - 1. Hence

$$\int p(u) d\mu_n(t;u) = a_n(t)\; p(1) + (1-a_n(t))p(-1)$$

where

$$0 \leq a_n(t) \leq 1$$

Hence

$$\int_\Omega f(t;u) d\mu_n = \int_0^1 a_n(t) f(t;1) dt$$

$$+ \int_0^1 (1-a_n(t)) f(t;-1) dt$$

$$\rightarrow \int_0^1 a(t) f(t;1) dt$$

$$+ \int_0^1 (1-a(t)) f(t;1) dt$$

Hence the limiting measure μ is such that

$d\mu(t;u)$ has a jump at $+1$ of $a(t)$

and a jump at -1 of $(1-a(t))$

Now

$$\int_0^1 \int_U u \, d\mu_n(t,u) \longrightarrow \int_0^1 a(t)dt - \int_0^1 (1-a(t))dt$$

Hence

$$\int_0^1 a(t)dt = \frac{1}{2}$$

Also

$$\int_\Delta \int_U u \, d\mu_n(t,u)$$

$$= (+1) \int_\Delta a_n(t)dt + (-1) \int_\Delta (1-a_n(t))dt$$

$$= \int_\Delta u_n(t)dt$$

$$\longrightarrow 0$$

Hence we must have:

$$a(t) = \frac{1}{2}$$

Hence

$$p[u_n(t)] \longrightarrow \int_U p(u)\, d\mu(t;u)$$

$$= \frac{1}{2}\, p(1) + (1 - \frac{1}{2}) p(-1)$$

Thus the limiting measure is a "chattering" between the values 1 and -1 with equal probability. Note that

$$u_n(t)^2 \longrightarrow \frac{1}{2} + (1 - \frac{1}{2}) = 1$$

which is correct.

Generalization is fairly transparent at this stage. For example, for the extension to the immediate case

$$u_n(t) = \text{one of } m \text{ values}, \quad u_1, \ldots u_m$$

and

$$u_n(\cdot) \text{ converges weakly to zero}$$

we have:

$$\int_U p(u)\, d\mu_u(t;u) \longrightarrow \int_U p(u) d\mu(t;u) = \sum_1^m a_k(t) p(u_k)$$

To determine the functions $a_k(t)$, we may note that

$$\sum_1^m a_k(t) = 1, \quad a_k(t) \geq 0$$

$$\sum_1^m a_k(t)\, u_k = 0$$

$$\sum_1^m a_k(t) u_k^2 = \text{limit} \quad u_k(t)^2$$

$$\sum_1^m a_k(t) u_k^{m-1} = \text{limit} \quad u_k(t)^{m-1}$$

giving us m equations to determine the m unknowns. The length of the time interval, so long as it is finite, obviously plays no role.

The weak limit of "ordinary controls" thus leads to a generalized control. Conversely, we have the following important result due to Young: Any generalized control can be approximated in the weak star topology of linear functionals on $C(\Omega)$ by ordinary controls. [Ordinary controls are weak-star dense in the class of generalized controls.]

[Of course the weak-star limits of generalized controls are quite obviously generalized controls.]

2. The Basic Technique

Let us illustrate our technique with reference to a simple control problem:

$$\text{Minimize: } \int_0^T g(t;x(t);u(t))dt$$

where

$$\dot{x}(t) = f(t;x(t);u(t)); \quad x(0) = x_o$$

and the control $u(t)$ is constrained to be in a restricted class of functions (called 'admissible' controls). We replace this problem by the non-dynamic epsilon problem:

$$\epsilon > 0$$

Minimize:

$$\frac{1}{2\epsilon} \int_0^T \|\dot{x}(t) - f(t,x(t),u(t))\|^2 dt + \int_0^T g(t,x(t),u(t))dt$$

over the class of state functions $x(t)$, absolutely continuous with $x(0) = x_o$ and the class of admissible controls. We present a constructive technique for solving this problem which as ϵ goes to zero approximates the original problem as closely as desired. The construction exploits the maximum principle indirectly; in fact the Hamiltonian arises in a natural way in the process.

See [2] for the bibliography and related work.

A Basic Estimate

We begin with the immediate question: how well does the epsilon problem approximate the original control problem? This question is of course of primary importance for computation, and it is interesting that we can answer it without the need for any of the usual assumptions of control thoery, even including the conditions that assure unique solution to the differential equation. We can also consider as general a class of control problems as necessary. However, in order not to confuse the main ideas with too much generality, we shall confine ourselves to the following class of problems (the extension to more general problems involving other types of phase plane constraints being readily made):

$$\text{Minimize} \quad \int_0^T g(t;x(t);u(t))\,dt \tag{2.1}$$

subject to:

$$\dot{x}(t) = f(t;x(t);u(t)) \quad \text{a.e.} \tag{2.2}$$

$$\phi(t;x(t);u(t)) = 0 \quad \text{a.e.} \tag{2.3}$$

where x(t) is absolutely continuous and satisfying additional conditions at the end points $t = 0$, and $t = T$. The end-point T is finite but of course not necessarily fixed. The control u(t) is Lebesgue measurable and subject to additional constraints, if any. We shall refer to such controls as "admissible" controls. It should be noted that not every admissible control necessarily yields a trajectory x(t) satisfying all the conditions, (2.2), (2.3) and the end conditions. However it would be natural to assume that there do exist admissible controls that lead to such trajectories. (Even this condition can be dispensed with for our purposes in this section.) Nor shall we need to impose any smoothness conditions on the functions $f(.)$, $g(\cdot)$ and $\phi(.)$. We shall only assume that they are Lebesgue measurable and such that the integral in (2.1) is well-defined for each (finite) T.

The epsilon problem is now formulated as follows:

Let

$$h(\epsilon;x(\cdot);u(\cdot);T) = \frac{1}{2\epsilon}\int_0^T ||\dot{x}(t) - f(t;x(t);u(t))||^2 dt + \frac{1}{2\epsilon}\int_0^T ||\phi(t;x(t);u(t))||^2 dt + \int_0^T g(t;x(t);u(t))\, dt \quad \ldots\ldots \quad (2.4)$$

Minimize $h(\epsilon;x(\cdot);u(\cdot);T)$ over the class of (absolutely continuous) trajectories x(t) subject to the given end conditions (any other "phase plane" constraints can clearly be added); and admissible controls u(t). We add the condition [F]:

$$||\dot{x}(t) - f(t;x(t);u(t))||^2 + ||\phi(t;x(t);u(t))||^2 \leq m < \infty$$

where m is a fixed positive constant independent of epsilon. This condition is not necessary if for example:

$$\text{Inf} \quad g(t;x;u) > -\infty \tag{2.5}$$

(as in time-optimal problems, see section 4)

The condition (F) is certainly a natural one in that we are, after all, trying to approximate the case $m = 0$. The need for such a condition may be seen by considering the simple example:

$$\dot{x}(t) = u(t) \; ; \; x(0) = 0 \quad ; \quad |u(t)| \leq 1$$

Minimize:

$$\int_0^1 [u(t)^2 - x(t)^4] \, dt$$

Here the epsilon problem without the finiteness condition will have minus infinity for the infimum while the control problem has zero for the infimum. Unless otherwise stated, this condition will be part of the epsilon problem in what follows.

Again in order not to complicate the exposition too much, we shall assume that the infimum of the epsilon problem is attained by a finite final time T_ϵ, in the sense that

$$h(\epsilon) = \text{Inf} \; h(\epsilon;x(\cdot);u(\cdot);T) = \lim_n \; h(\epsilon;x_n(\cdot);u_n(\cdot);T_\epsilon)$$

where T_ϵ is finite, and $x_n(\cdot)$, $u_n(\cdot)$, is a "minimizing" sequence for the epsilon problem. For such a minimizing sequence, let

$$d(\epsilon) = \text{Lim inf } \frac{1}{2}\int_0^{T_\epsilon} (||\dot{x} - f(t;x_n(t);u_n(t))||^2 + ||\phi(t;x_n(t);u_n(t)||^2)dt$$

$$G(\epsilon) = \lim \sup \int_0^{T_\epsilon} g(t;x(t);u(t))\, dt$$

Then of course

$$h(\epsilon) = d(\epsilon)/\epsilon + G(\epsilon)$$

Let us now define

$$\delta(\epsilon) = \text{Sup } d(\epsilon)$$

$$g(\epsilon) = \text{Inf } G(\epsilon)$$

where the infimum (and supremum) is taken over the class of all minimizing sequences. While $\delta(\epsilon)$ is finite because of condition (F), $g(\epsilon)$ may well be minus infinity in general. Under the usual conditions on the dynamics, we shall see however that $g(\epsilon)$ will be finite. We have of course:

$$h(\epsilon) = \delta(\epsilon)/\epsilon + g(\epsilon) \tag{2.6}$$

It is natural now to define $g(0)$ to be the infimum for the control problem, assuming it is definable. Then

$$h(\epsilon) \leq g(0) \tag{2.7}$$

With these definitions we can state the following theorem concerning the approximation:

Theorem 2.1

Suppose $g(\epsilon)$ is finite for some ϵ_o. Then $g(\epsilon)$ is finite for every ϵ less than ϵ_o, and moreover as $\epsilon \to 0$, $\delta(\epsilon)$ is monotone non-increasing and $g(\epsilon)$ is monotone non-decreasing.

Further:

$$\lim_{\epsilon \to 0} \delta(\epsilon)/\epsilon = 0 \tag{2.8}$$

if $g(0)$ is definable (not equal to plus infinity).

Proof

Let ϵ be less than ϵ_o. We have then, as an elementary analysis on sums of limits shows:

$$\delta(\epsilon)/\epsilon + g(\epsilon) = h(\epsilon) \leq \delta(\epsilon_o)/\epsilon + g(\epsilon_o)$$

and similarly:

$$\delta(\epsilon_o)/\epsilon_o + g(\epsilon_o) = h(\epsilon_o) \leq \delta(\epsilon)/\epsilon_o + g(\epsilon)$$

Since every quantity is finite on the left we can freely transpose to obtain:

$$\frac{\delta(\epsilon) - \delta(\epsilon_o)}{\epsilon} \leq g(\epsilon_o) - g(\epsilon) \leq \frac{\delta(\epsilon) - \delta(\epsilon_o)}{\epsilon_o} \qquad (2.9)$$

and since ϵ is less than ϵ_o, these relations are consistent only if

$$\delta(\epsilon) \leq \delta(\epsilon_o)$$

$$g(\epsilon) \geq g(\epsilon_o)$$

Hence $g(\epsilon)$ is finite. Moreover since the argument can now be repeated with

$$\epsilon = \epsilon_1, \qquad \epsilon_o = \epsilon_2, \qquad \epsilon_1 < \epsilon_2 \leq \epsilon_o$$

the required monotonicity follows. Let $g(0+)$ denote the limit of $g(\epsilon)$ as ϵ goes to zero. From (2.7), since $g(0)$ is not plus infinity, $\delta(\epsilon)$ must converge to zero. Again with $\epsilon < \epsilon_1 < \epsilon_o$, we have

$$g(\epsilon) - g(\epsilon_1) \geq \frac{\delta(\epsilon_1) - \delta(\epsilon)}{\epsilon_1}$$

and letting ϵ go to zero in this we obtain that for $\epsilon < \epsilon_o$,

$$\delta(\epsilon)/\epsilon \leq g(0+) - g(\epsilon) \leq g(0) - g(\epsilon) < \infty \tag{2.10}$$

and in particular then $\delta(\epsilon)/\epsilon$ goes to zero.

Remark 1

It should be noted that infimum of the epsilon problem has been sought in the class of admissible controls. This is natural since, freed of having to satisfy the differential equation constraint, any admissible control can be used. On the other hand this means that in general the optimal control will be a relaxed control. In particular g(0+) may well be less than g(0), the latter being usually sought in the class of ordinary controls, as we assume herein also. An example is given in [3] where $g(0+) = -1$, while $g(0) = 0$. However we shall see that the infimum for the control problem allowing relaxed controls will be $g(0+)$, at least under the usual conditions. But the main point is that in the epsilon problem relaxed controls appear of necessity.

Remark 2

As shown in [3], (2.9) and (2.10) actually hold for $d(\epsilon)$ and $G(\epsilon)$ (even though the latter may depend on the particular minimizing sequence chosen!)

<u>Corollary</u> Assume $g(0) < +\infty$ and that $g(\epsilon_o)$ is finite for some $\epsilon_o > 0$. Then, $h(\epsilon)$ is monotone non-decreasing and omitting at most a countable number of points in $0<\epsilon<\epsilon_o$, we have

$$h'(\epsilon) = (-1)\delta(\epsilon)/\epsilon^2 \tag{2.11}$$

and

$$g'(\epsilon) + \delta'(\epsilon)/\epsilon = 0 \quad \text{a.e. in } 0<\epsilon<\epsilon_o . \tag{2.12}$$

<u>Proof</u> For $\epsilon<\epsilon_o$, both $g(\cdot)$ and $\delta(\cdot)$ are monotone, and hence continuous except for a countable number of points and differentiable a.e. Now

$$h(\epsilon + \Delta) - h(\epsilon) \leq (\delta(\epsilon)/(\epsilon + \Delta) + g(\epsilon)) - (\delta(\epsilon)/\epsilon + g(\epsilon))$$

$$= \delta(\epsilon)\,(1/(\epsilon + \Delta) - 1/\epsilon)$$

(showing monotonicety)

while

$$h(\epsilon + \Delta) - h(\epsilon) \geq \left(\frac{\delta(\epsilon + \Delta)}{\epsilon + \Delta} + g(\epsilon + \Delta)\right) - (\delta(\epsilon + \Delta)/\epsilon + g(\epsilon + \Delta))$$

$$= \delta(\epsilon + \Delta)\left(\frac{1}{(\epsilon + \Delta)} - \frac{1}{\epsilon}\right)$$

or, (2.11) follows. But omitting a set of measure zero:

$$h'(\epsilon) = \delta'(\epsilon)/\epsilon + g'(\epsilon) - \delta(\epsilon)/\epsilon^2$$

from which (2.12) follows.

3. Fixed End-Point Problems

In order to introduce the basic ideas in the epsilon technique, it is convenient to begin with what is perhaps the simplest class of control problems: Fixed end-point problems with fixed initial condition, and bounded controls.

Problem I:

$$\text{Minimize } \int_0^T g(t;x(t);u(t))\,dt \quad + \varphi(x(T)) \tag{3.1}$$

where T is fixed and finite and

$$\dot{x}(t) = f(t;x(t);u(t)) \quad \text{a.e.} \ ; \quad x(0) = x_1 \text{ fixed} \tag{3.2}$$

$u(t)$ Lebesgue measurable

$$u(t) \in U \quad \text{a.e.}, \quad U \text{ being compact} \tag{3.3}$$

It will be assumed in addition that

$f(t;x;u)$, $g(t;x;u)$, $\varphi(x)$ are C^1 in x, continuous in all variables, and further condition G holds: +

$$\text{(G):} \qquad [x, f(t;x;u)] \leq c(1+\|x\|^2) \quad \text{for } u \text{ in } U, \ 0 \leq t \leq T \cdots \tag{3.4}$$

We note immediately that the infimum, denoted $g(0)$, is finite. The epsilon problem is formulated as follows:

Let

$$h(\varepsilon;x(\cdot);u(\cdot) = \frac{1}{2\varepsilon}\int_0^T \|\dot{x} - f(t;x(t);u(t))\|^2\,dt + \varphi(x(T))$$

$$+ \int_0^T g(t;x(t);u(t))\,dt \tag{3.5}$$

+ This condition as well as [3.3] can be relaxed as in [15] for example - we forego this generalization in the interest of simplicity of exposition, especially since it is not an intrinsic limitation on the approach.

Minimize $\dot{h}(\epsilon;x(\cdot);u(\cdot))$ over the class of controls $u(t)$ Lebesgue measurable, $u(t) \in U$, and also over the class of absolutely continuous ('state') functions $x(t)$ with $x(0) = x_1$. (It is clear that additional phase plane constraints can be added here if necessary.) In addition the condition F' is imposed:

$$(F'): \qquad \int_0^T \|\dot{x}(t)\|^2 \, dt \leq m < \infty \tag{3.6}$$

The condition F' is a slight weakening of condition F, which is possible because of the smoothness properties of the functions assumed. Thus let $x_n(\cdot)$, $u_n(\cdot)$ be a minimizing sequence for the epsilon problem. Condition F' implies that $x_n(t)$ is uniformly bounded in $0 \leq t \leq T$ and hence both $\delta(\epsilon)$ and $g(\epsilon)$ (which now includes the $\varphi(\cdot)$ term) are finite. Again, it is readily seen that condition F implies F'. For let

$$\dot{x}_n - f(t;x_n(t);u_n(t)) = z_n(t) \tag{3.7}$$

Then, using (3.4):

$$\begin{aligned} |[\dot{x}_n, x_n]| &\leq |[x_n, f(t;x_n(t);u_n(t))]| + |[x_n, z_n]| \\ &\leq c(1 + \|x_n\|^2) + m\|x_n\| \\ &\leq D(1 + \|x_n\|^2) \end{aligned} \tag{3.8}$$

and by the usual analysis (Gronwall lemma) this implies that $x_n(t)$ is uniformly bounded. (If the initial condition $x_n(0) = x_1$ is generalized to $\varphi_o(x(0) = 0$, we must then require that the set

$$[x \mid \varphi_o(x) = 0]$$

is bounded, for this result to hold as well as for $g(0)$ to be finite.)

To solve the epsilon problem we take the following elementary route. Let an admissible state function $x(t)$ (that is, absolutely continuous and satisfying $x(0) = x_1$ and (3.6)) be chosen. To minimize (3.5), we simply minimize the integrand. Let

$$m(\varepsilon;t;y;x) = \underset{u \, \varepsilon \, U}{\text{Min}} \left(\frac{1}{2\varepsilon} \| y - f(t;x;u) \|^2 + g(t;x;u) \right) \tag{3.9}$$

The minimum is clearly attained since U is compact and the functional is continuous. It is readily seen further that $m(\varepsilon;t;y;x)$ is continuous in all the variables. Now

$$h(\varepsilon;x(\cdot);u(\cdot)) \geq \int_o^T m(\varepsilon;t;\dot{x}(t);x(t))\,dt + \varphi(x(T))$$

so that

$$h(\varepsilon) \geq \text{Inf} \int_o^T m(\varepsilon;t;\dot{x}(t);x(t))\,dt + \varphi(x(T))$$

where the infimum is taken over the class of admissible state functions $x(t)$. To reverse the inequality in (3.10) we have only to note that we can find an admissible control u(t) such that (a.e.):

$$m(\epsilon;t;\dot{x}(t);x(t)) = \frac{1}{2\epsilon} \| \dot{x}(t) - f(t;x(t);u(t)) \|^2 + g(t;x(t);u(t))$$

This is obvious if the minimum of (3.9) is attained at a unique point in U. Otherwise we invoke the "half-way principle of McShane and Warfield", as in Young [1].

Let $x_n(\cdot)$ be a minimizing sequence for

$$h(\epsilon) = \text{Inf} \int_0^T m(\epsilon;t;\dot{x}(t);x(t))dt + \varphi(x(T))$$

Let $u_n(\cdot)$ be a corresponding admissible control sequence. Now it is readily seen that $x_n(\cdot)$ is equicontinuous. Hence we may, by renumbering if necessary, assume that $x_n(t)$ converges uniformly to $x_o(t)$ say. Further we can see that $x_o(t)$ is absolutely continuous and we may assume that the sequence (again by renumbering as necessary) $\dot{x}_n(t)$ converges weakly to $\dot{x}_o(t)$. Also $x_o(t)$ is an admissible state function. But the sequence of controls converge, in general, only in the sense of relaxed controls. [Indeed to establish the existence of a relaxed optimal control for the epsilon problem, as in the original control problem, takes "no more than a routine exercise in using the Ascoli theorem and the diagonal process" (McShane [4]) only more so in the present case!]