C. Cattaneo (Ed.)

CIME Summer Schools

Relatività generale

32

Salice d´Ulzio, Italy 1964

Springer

FONDAZIONE CIME ROBERTO CONTI

C. Cattaneo (Ed.)

Relatività generale

Lectures given at the
Centro Internazionale Matematico Estivo (C.I.M.E.),
held in Salice d´Ulzio (Torino), Italy,
July 16-25, 1964

Springer

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ISBN 978-3-642-11020-7 e-ISBN: 978-3-642-11021-4
DOI:10.1007/978-3-642-11021-4
Springer Heidelberg Dordrecht London New York

Reprint of the 1st ed. C.I.M.E., Ed. Cremonese, Roma, 1965
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Printed on acid-free paper

Springer.com

CENTRO INTERNATIONALE MATEMATICO ESTIVO
(C.I.M.E)

Reprint of the 1st ed.- Salice d´ Ulzio, Italy, July 16-25, 1964

RELATIVITÀ GENERALE

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P R E F A C E

The following lectures were intended to serve as an introduction to the theory of gravitational waves, mainly for mathematicians not specialized in the field of general relativity. Accordingly, basic concepts and motivations an the purely local, differential geometrical "pure" radiation theory have been put in the foreground, and conceptually and computationally more complicated recent advances have indicated only briefly.

The references and footnotes should be considered an essential part of the course; I hope that some of them serve to clarify points raised in discussions which followed the lectures.

GRAVITATIONAL WAVES

by Jürgen Ehlers

1. Introduction : The Basis of the General Theory of Relativity

From a physicist's point of view the general theory of relativity is of basic importance, despite of its very poor experimental or observational verification, for two reasons :

a) It is the most convincing field theory of gravitation which is locally compatible with the experimentally well-established Lorentzian structure of the space-time metric, and

b) it is the most important example of a physical theory in which the metric structure of space-time is treated not as given a priori, but dependent on and interrelated to other physical variables describing processes in space-time.

Although b) is not independent of a) it is worthwhile to stress the autonomous importance of aspect b): So far, every physical theory, whether non-relativistic or relativistic, classical or quantum, whether a particle -or a field theory, requires for the formulation of its basic laws as well as for its interpretation a metric and, associated with it, an affine connection which serves to formulate laws relating quantities with directional properties at different space-time points or "events". This implies that in all physical theories the metric has a strong influence on other physical quantities - I need only mention the law of inertia so fundamental not only for classical mechanics but also for, say, the quantum theory of scattering. Nevertheless this metric structure is not reinfluenced by these physical quantities except in the general theory of relativity and its generalizations. This strongly suggests the idea that the pre-Einsteinian theories may well be considered as approximate theories which describe situations in which the metric field can be treated as an external field which has, under the special circumstances considered, always the same structure, whereas in more general cases or in a more precise description the metric is a field variable like, say, the electromagnetic field. It is certainly more convincing to have a theory where all quantities which are used to interpret the observed phenomena are interrelated ("principle of omnipresence of all state variables", to use a phrase from the modern theory of irreversible processes in continuous media) than the assign some of these quantities a priori and prescribe "laws" only for the remaining ones.

If this point of view is accepted, then the gravitational field - if it is identified with the metric field - acquires, despite of its extreme weakness even in comparison with so called "weak" interactions, a fundamental role in physics since it is coupled to all other fields, due to the role of the metric stressed above. There is a very good reason for this identication, namely the universal proportionality of inertial and ("passive") gravitational mass of bodies substantiated with a precision of 10^{-11} by the Eötvös-Dicke experiment[1].

Let us, then, accept this idea of the metric as a physical field, and formulate the first basic assumption of the Einsteinian theory, motivated by the special theory of relativity :

(G) Riemannian assumption : The space-time manifold $V_4 = V$ carries a normal-hyperbolic Riemannian metric with the fundamental quadratic form (in an arbitrary local coordinate-system)

$$G = g_{ab}(x^c)dx^a dx^b \quad . \qquad (1 \leqslant a, b, \dots \leqslant 4)$$

(We take the signature to be +++-.)

Since g_{ab} is supposed to describe the gravitational field well-known considerations of Einstein (which contain some weak points which are still not clarified completely) lead to the assumption that the source of the g_{ab} -field is the stress-energy-momentum tensor of all the matter populating V ; we formulate the second assumption :

(T) The mechanical properties of matter are described by a symmetric tensor field T_{ab} depending on the state-variables of matter.

Finally, Newtonian theory (Poisson's equation), simplicity-requirements (quasi-linearity, second differentiation order for the g_{ab} 's) and energy-momentum conservation[2] suggest the most specific assumption of Einstein's 1915-theory :

(GT) The metric G of space time is related to the stress energy momentum distribution T in V by the field equation[3]

$$G_{ab} + T_{ab} = 0 \quad . \tag{1}$$

Here the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ occurs where R_{ab} is the contracted curvature tensor, and R its trace. (We choose units such that $c = 1$ and (Newton's constant of gravity) $= \frac{1}{8\pi}$.)

The gravitational field equation (1) is, of course, not sufficient as a basis for a theory of the interaction between matter and the gravitational field. It is necessary to add assumptions about the structure of matter, i.e. to specify the dependence of T_{ab} on the basic matter (or field) variables, and to state the non-gravitational equations of motion which these variables are supposed to obey. Since, however, the interaction between matter and gravitational waves has so far not been investigated in the full, non-linear theory, we need not specify such assumptions here.

In empty space, where (1) reduces to

$$G_{ab} = 0, \quad \text{or} \quad R_{ab} = 0 \quad , \tag{2}$$

no further assumptions are needed in order to calculate the time-development of a field from given initial data.

The gravitational field equation implies the "mechanical law"

$$T^{ab}{}_{;b} = 0 \tag{3}$$

which is the general-relativistic analogue of the balance-equations for energy and momentum for continuous media.

For isolated bodies of appropriate internal structure one can "approximately deduce" from (3) equations of motion for the center of mass world line and for multipole moments describing the structure of the body such as spin, quadruple moment etc. We adopt here[4] as equations of motion of an approximately rigid, spherically symmetrical test particle with internal angular momentum per proper mass S^a :

$$\frac{dx^a}{ds} = u^a , \quad \frac{\nabla u^a}{ds} + R^{*a}{}_{bcd} u^b u^d S^c = 0 \quad , \tag{4$_1$}$$

$$\frac{\nabla S^a}{ds} (\delta^b_a + u_a u^b) = 0 \quad . \tag{4$_2$}$$

s denotes the proper time, u^a the 4-velocity, $u_a u^a = -1$, $\frac{\nabla u^a}{ds}$ is the 4-acceleration, and

$$R^{*\,abcd} = \frac{1}{2} R^{ab}{}_{ef} \eta^{efcd}$$

is the "right-dual" of the Riemann curvature tensor. The metric quantities are those of the external field.

A nonspinning test particle has, according to $(4)_1$, a geodesic world line. For a pair of neighbouring nonspinning test particles the relative acceleration is a linear transform of the relative position vector x^a , $u_a \delta x^a = 0$:

(5)
$$\frac{\nabla^2 \delta x^a}{ds^2} = R^a{}_{bcd} u^b u^d \delta x^c$$

These equations of motion for test particles give direct operational meaning to the metric g_{ab} since the set of all timelike geodesics determines a normal hyperbolic metric uniquely up to a constant factor[5]. Moreover, $(3)_1$ and (4) give a precise meaning to the statement that "the curvature tensor describes the strength and the directional properties of a gravitational field similarly to the way in which the field strength tensor describes an electromagnetic field.

Finally, we observe that $(4)_2$ gives a physical meaning to the Fermi propagation of vectors along curves. Since we may take the spin as small as we like[6] for a given mass, we can, to any desired degree of accuracy, realise a geodesic with a vector parallely propagated along it and orthogonal to the curve. Taking two such test-gyroscopes near one another, we can supplement (5) by the statement[7]:
The difference δS^a between tha angular momentum of the first particle and that of the second particle parallel displaced along the connection vector δx^a and projected into the local space orthogonal to the 4-velocity u^a of the first particle, $\delta_\perp S^a$, obeys the law

$(6)_1$
$$\frac{\nabla \overrightarrow{\delta_\perp S^a}}{ds} = \vec{S} \times \vec{H} ,$$

where

$(6)_2$
$$H^a = R^{*a}{}_{bcd} u^b u^d \delta x^c .$$

Here $\delta_{\perp} S^a$, S^a , H^a belong to the 3-space orthogonal to u^a , and $(6)_1$ is written as a 3-vector relation containing the usual exterior product. Rewritten in this notation, (5) assumes the form

$$(5)_1 \qquad \frac{\nabla^2 \overrightarrow{\delta x}}{ds^2} = \vec{E} \quad ,$$

$$(5)_2 \qquad E^a = R^a{}_{bcd} u^b u^d \delta x^a \quad .$$

The equations (6) and (5') exhibit that, for a given "observer" u^a , the (spatial) vectorfields $\vec{E}$ and $\vec{H}$ defined in the infinitesimal neighbouhood of the observer's world line play a similar role for a gravitational field as the electric and magnetic vectors relative to an inertial frame for an electromagnetic field.

A null-geodesic also has a physical interpretation : It represents the world line of a particle of vanishing rest mass or, more classically, a light ray in the sense of geometrical optics. In this case, the statement can be "approximately deduced" by starting with the general relativistic form of Maxwell's equations and going over to the limit of "locally plane waves of infinitely small wave-length"[8].

As long as we do not have a description of the interaction of matter with gravitational fields, especially gravitational waves, the preceding remarks on test-body motions are a useful preliminary tool for the physical interpretation of algebraic and anlytic properties of vacuum gravitational fields and, especially, their curvature tensors. One should keep in mind, however, that this description of the action of gravitational fields on matter is very incomplete since the reaction of the particles on the fields is completely neglected.

2. The linear approximation. Survey of problems

In order to get a survey over the problems with which we are faced let us at first drastically eliminate the mathematical complications due to the non-linearity of eqs. (1) :

Let us denote by Δ_{ab} the orthonormal components of the flat space time metric, and let us assume that the quantities

$$\eta_{ab} \equiv g_{ab} - \Delta_{ab} \tag{7}$$

satisfy the "weak field conditions"

$$|\eta_{ab}| \ll 1 \quad , \tag{8$_1$}$$

$$\left|\Gamma^{e}_{b[c}\Gamma^{a}_{d]e}\right| \ll \left|\Gamma^{a}_{b[c,d]}\right| \quad , \tag{8$_2$}$$

where Γ^{a}_{bc} are the Christoffel symbols associated with the g_{ab} . Then the field equation (1) reduces, in the sense of a formal approximation in which small quantities are neglected, to the linearised field equation

$$\Box\,\psi_{ab} - 2\psi_{(a,b)} + \Delta_{ab}\,\psi^{c}{}_{,c} = -2T_{ab} \tag{9}$$

here

$$\psi_{ab} \equiv \eta_{ab} - \frac{1}{2}\Delta_{ab}\eta \quad , \quad \eta \equiv \eta^{a}{}_{a}, \quad \psi_{a} \equiv \psi_{a}{}^{b}{}_{,b}, \tag{10}$$

and the D'Alembert-operator $\Box$ and the raising and lowering of indices refer to the flat metric Δ_{ab} .

A "small" coordinate change

(11) $$x^{a'} = x^a + \xi^a \quad , \quad \left| \xi^a{}_{,b} \, \xi^c{}_{,d} \right| \ll \left| \xi^e{}_{,f} \right|$$

induces the transformation

$(12)_1$ $$\psi_{a'b'} = \psi_{ab} - 2\,\xi_{(a,b)} + \Delta_{ab}\,\xi^c{}_{,c} \quad ,$$

$(12)_2$ $$\psi_{a'} = \psi_a - \Box\,\xi_a$$

of the field variables ψ_{ab} .

One may now forget the "derivation" of (9) and (12) from the rigorous theory and consider (9) as a gravitational field equation in flat space-time, formally very similar to electrodynamics. Then (12) can be considered not as induced by a coordinate transformation bur as a gage-transformation; in fact, the substitution (12) (with unchanged independent variables x^a !) leaves the left hand side of eq. (9) unchanged. It also follows from (9) that

(13) $$T^{ab}{}_{,b} = 0 \quad .$$

But this equation clearly shows this linear theory of gravitation being physically wrong : According to (13), the gravitational field ψ_{ab} would have no influence on the energy and momentum balances of matter. Although the field is determined by its source T_{ab} only up to gage transformations the linearised equation of motion of a test particle is not gage invariant; this is a second inconsistency[9].

We therefore have to consider (9) at best as the first step in a sequence of successive approximations[10]. Let us nevertheless apply the flat-space interpretation of (9), (12) in the following and state some mathematical properties of this theory rigorously, as a motivation for the analysis of the full theory.

Let us consider a spatially bounded source T^{ab} at rest in some inertial frame. Then the retarded integral

$$\Psi_{ab}(x) = \frac{1}{2\pi}\int_{C_x^-} T_{ab}\, dK \tag{14}$$

exists and satisfies, if (13) holds, the Einstein convention

$$\Psi_a = 0 \tag{15}$$

and the field equation (9). (dK is the Lorentz-invariant measure on the past light cone C_x^- of x.) If T^{ab} and its first derivatives are bounded in the past, (14) satisfies the boundary conditions

$$\Psi_{ab} = \theta\left(\frac{1}{r}\right), \qquad \Psi_{ab,c} = \chi_{ab}k_c + \theta\left(\frac{1}{r}\right),$$

$$\chi_{ab} = \theta\left(\frac{1}{r}\right), \qquad \Psi_a = \theta\left(\frac{1}{r^{z+\varepsilon}}\right), \qquad \varepsilon > 0 .$$

r denotes the spatial distance of the argument of $\Psi_{..}$ from a time like straight line contained in the source region, and k^a is a null vector field pointing away from the source and into the future, normalised according to

$k^a u_a = -1$ if u^a is the 4-velocity of the line mentioned above.

We now state the

theorem [11] : For a given source T^{ab} exists up to gage transformations one and only one solution ψ_{ab} of (9) which satisfies the "outgoing radiation condition" (16). Among these, precisely one satisfies the Einstein convention (15) .

To prove uniqueness, we apply the Kirchhoff integral representation[12] (known in physics from the theory of diffraction)

$$(17) \qquad 4\pi \cdot \psi_{ab}(x) = -\int_{C_x^-} \Box \psi_{ab} \, dK$$

to the difference of two solutions of (9) both satisfying (16). The surface integrals in the general Kirchhoff-representation can and have been shifted to (past) infinity in C_x^- and then give zero because of $(16)_{1,2,3}$. From (9) and (17) we obtain for this difference $4\pi \psi_{ab}(x) = -\int_{C_x^-} (2 \psi_{(a,b)} - \Delta_{ab} \psi^c{}_{,c})d$

which can be written, on account of $(16)_4$, in the form $-2 \xi_{(a,b)} + \Delta_{ab} \xi^c{}_{,c}$ with $\xi_a(x) = \int_{C_x^-} \psi_a \, dK$, and is, consequently, gage-equivalent to zero. The existence has already been shown.

A motivation for the name "outgoing radiation condition" for (16) can be seen in the fact that the change

$$d \psi_{ab} = \chi_{ab} k_c dx^c + \theta(\frac{1}{r^2})_c \, dx^c$$

at large distances from the source is smallest for displacements within the hypersurfaces of constant phase, $k_a dx^a = 0$.

Because of this theorem, it is no loss of generality for problems involving bounded sources only to impose generally the condition (15), i.e.

(18) $$\psi_{a\,;\,b}^{\;b} = 0 \;.$$

Then (9) simplifies to

(19) $$\Box\,\psi_{ab} = -2T_{ab}$$

Outside of the sources, we have

(20) $$\Box\,\psi_{ab} = 0, \qquad \psi_{a\;,\,b}^{\;b} = 0 \;,$$

and we define, in the linear approximation, "free" gravitational waves as gage-equivalente classes of solutions of (20).

The "free" classical field theory defined by (20) can be used to construct a corresponding special relativistic quantum theory of a "graviton field" . For this purpose one hase to define, on a suitably chosen subset of the solutions of (20), a Hilbert space structure with a scalar product that is invariant under (inhomogeneous) Lorentz transformations. You obtain thus an irreducible unitary representation of the inhomogeneous Lorentz group in a Hilbert space of solutions of (20) which is to be interpreted physically as the space of one - graviton states. According to the group theoretic classification of fundamental particles (or fields), one then finds the linearised free graviton field belonging to particles with vanishing rest mass and spin 2. (The spaces of n-particle states and, finally, the total (Fock-) space of the free graviton field can be constructed by standard procedures from the space of one-particle states.)

The metric corresponding to the general solution of (20) which represents a plane wave travelling in the z-direction can be written in the form

$$G = \underset{0}{G} + A(dx^2 - dy^2) + 2B\,dxdy \tag{21}$$

with two arbitrary functions A, B of the "phase" $u = z - t$. $\underset{0}{G}$ denotes the Minkowskian metric.

Since the coefficients of (21) depend on u only, the curvature tensor (and all intrinsic characteristics of the metric field) are propagated without change along the rays $(x, y, u) = \text{const.}$ which form a congruence of null geodesics, i.e. a plane gravitational wave propagates without distortion with fundamental velocity.

(21) is in Gauss'normal form with respect to t, and thus the geodesics $(x, y, z) = \text{const.}$ may bethought of as world lines of test particles. It follows from (21) that a cloud of such particles undergoes a volume-preserving deformation which is restricted to directions orthogonal to the direction of propagation of the wave; the magnitude of this deformation depends on the amplitudes A, B.

In order to characterize the wave (21)independently of a special set of test particles we use the linearized curvature tensor. It has the form

$$R_{abcd} = m_{ab}m_{cd} - \overset{*}{m}_{ab}\overset{*}{m}_{cd} \tag{22_1}$$

where m_{ab} is a singular bivector,

$$m_{ab}m^{ab} = 0 \quad , \quad \overset{*}{m}_{ab}m^{ab} = 0 \quad . \tag{22_2}$$

Consequently there exists a null vector k^a such that

$$(22)_3 \qquad m_{ac} m_b{}^c = k_a k_b \qquad (\Longrightarrow \; m_{ab} k^b = 0).$$

The quantities m_{ab} , k^a are determined by R_{abcd} up to their signs.

The interpretation of m_{ab}, k^a follows from eq. (5) : Let u^a be the 4-velocity of an arbitrary nonspinning test particle or "observer" . Then

$$(23)_1 \qquad p^a \equiv \frac{m^a{}_b u^b}{-k_c u^c} \quad , \qquad q^a \equiv \frac{\overset{*}{m}{}^a{}_b u^b}{-k_c u^c}$$

form an orthogonal pair of (with respect to this observer) purely spatial vectors, and (22) may be rewritten as

$$(23)_2 \qquad R_{abcd} = 4\, k_{[b} i_{a]} {}_{[c} k_{d]} \; ,$$

$$(23)_3 \qquad i_{ab} \equiv p_a p_b - q_a q_b \quad .$$

The accelerations of nearby test particles relative to our observer are, according to (5) and (23) , given by

$$(24) \qquad \frac{\nabla^2 \delta x^a}{ds^2} = (k_c u^c)^2 \, i^a{}_b \, \delta x^b \quad .$$

These formulae show : The acceleration of a test particle relative to a freely falling observer vanishes if and only if its position vector δx^a is parallel to the projection $k^a_{\perp}$ of k^a into the observers 3-space. $(k_a u^a)^2$ is equal to the ratio (magnitude of relative acceleration / distance) for arbitrary nearby test particles. The acceleration is parallel to δx^a if $\delta x^a = \lambda p^a$, anti-parallel if $\delta x^a = \lambda q^a$.

For the wave (21) k^a is given by

$$(25) \qquad k_a dx^a = (\frac{1}{4}(A''^2 + B''^2))^{1/4} du \quad ,$$

and the propagation of the wave along the rays is expressed by

$$(26)_1 \qquad m_{ab;c} k^c = 0$$

which implies, by (22),

$$(26)_2 \qquad R_{abcd;e} k^e = 0, \qquad k_{a;b} k^b = 0 \quad .$$

A freely falling observer, however, will notice changes of the field; the strength $(k_a u^a)^2$ will be a function of this proper time, and the directions $k_\perp^a$, p^a, q^a will rotate relative to spatial axes which are parallely propagated along his world line. These changes may be used to define, with respect to an observer, monochromatic waves and, among them, linearly, circularly etc. polarized waves quite similar to electrodynamics.

We finally remark that vacuum curvature tensors of the algebraic type (22) can be characterized by the existence of a vector k^a such that

$$(27) \qquad R_{abcd} k^d = 0 \qquad (\Longrightarrow \quad k_a k^a = 0) \quad ;$$

this remark suggests a way of defining pure radiation fields in the rigorous theory.

Let us now return to the inhomogeneous equation (19) and its retarded solution (14). If we choose an inertial frame in which the source is at rest and located near the origin of the space-coordinates we may write

$$(28) \qquad \psi_{ab}(\underline{x}, t) = \frac{1}{2\pi} \int \frac{T_{ab}(\underline{y}, t - |\underline{x} - \underline{y}|)}{|\underline{x} - \underline{y}|} \, d^3y$$

($\underline{x}$, $\underline{y}$ denote 3-vectors). If we are interested in the radiation field at large distances from the source, we will, as usual, write $t - |\underline{x} - \underline{y}| =$ $= t - |\underline{x}| + (|\underline{x}| - |\underline{x} - \underline{y}|)$ and develop $\frac{1}{|\underline{x} - \underline{y}|}$ and $|\underline{x}| - |\underline{x} - \underline{y}|$ in powers of r^{-1}, obtaining

$$(29)_1 \qquad \psi_{ab}(\underline{x}, t) = \frac{N_{ab}(u, \omega)}{r} + \theta\left(\frac{1}{r^2}\right)$$

where we have written u for the retarded time $t - |\underline{x}|$, ω for the direction given by the unit vector $\frac{\underline{x}}{r}$, $r = |\underline{x}|$, and

$$(29)_2 \qquad N_{ab}(u, \omega) \equiv \frac{1}{2\pi} \int T_{ab}\left(\underline{y}, u + \frac{\underline{x} \cdot \underline{y}}{r}\right) d^3y \qquad .$$

From (29) it follows that

$$(30)_1 \qquad \psi_{ab;c} = \frac{-\dot{N}_{ab}}{r} k_c + \theta\left(\frac{1}{r^2}\right)$$

$$(30)_2 \qquad \psi_{ab;cd} = \frac{\ddot{N}_{ab}}{r} k_c k_d + \theta\left(\frac{1}{r^2}\right) \qquad ,$$

here the dot indicates a partial derivative with respect to the retarded time for fixed ω, and k^a is chosen as in (16). Since the Einstein convention

(15) is satisfied in consequence of (13) we also have

$$(30)_3 \qquad \dot{N}_{ab}k^b = \theta(\frac{1}{r}) \quad , \quad \Longrightarrow \ddot{N}_{ab}k^b = \theta(\frac{1}{r}) \quad .$$

$(30)_{2,3}$ give for the linearized curvature tensor the expression $\left\{ (23)_2 + \theta(\frac{1}{r^2}) \right\}$ with

$$(31) \qquad i_{ab} = (2r)^{-1} (\ddot{N}_{ab} - \frac{1}{2} \Delta_{ab} \ddot{N}^c{}_c) + \theta(\frac{1}{r^2}) \quad .$$

This result proves : The $\frac{1}{r}$ - part of the curvature tensor which belongs to the retarded radiation field of a bounded source has the same algebraic structure as that of a plane wave.

The development indicated before $(29)_1$ can of course be carried on further; the coefficients of the higher powers of r^{-1} will be functions of (u, ω) which can be represented by integrals like $(29)_2$ with T_{ab} replaced by its time derivatives, multiplied by polynomials in $\underline{y}^2$ and $\frac{\underline{x} \cdot \underline{y}}{r}$. We shall return to this result in the rigorous theory. If the changes withinthe source are sufficiently slow it is useful to develop the integrand of $(29)_2$ in powers of $\frac{\underline{x} \cdot \underline{y}}{r}$; this leads to a multipoleexpansion of the radiation field. Because of the energy-momentum conservation law (13) the lowest order radiation is of the quadrupole type.

We may finally ask : What is the energy carried away from a source by gravitational radiation? If we accept the gravitational energy tensor[20]

$$(32) \quad t_{ab} = \frac{1}{4} (\psi_{cd,a} \psi^{cd}{}_{,b} - \frac{1}{2} \psi_{,a} \psi_{,b} - \frac{1}{2} \Delta_{ab} \bar{\psi}_{cd,e} \psi^{cd,e} - \frac{1}{2} \psi_{,c} \psi^{,c})$$