G. Prodi (Ed.)

Eigenvalues of Non-Linear Problems

CIME Summer Schools

67

Varenna, Italy 1974

Springer

FONDAZIONE CIME ROBERTO CONTI

G. Prodi (Ed.)

Eigenvalues of Non-Linear Problems

Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
held in Varenna (Como), Italy,
June 16-25, 1974

Springer

FONDAZIONE CIME ROBERTO CONTI

C.I.M.E. Foundation
c/o Dipartimento di Matematica "U. Dini"
Viale Morgagni n. 67/a
50134 Firenze
Italy
cime@math.unifi.it

ISBN 978-3-642-10939-3 e-ISBN: 978-3-642-10940-9
DOI:10.1007/978-3-642-10940-9
Springer Heidelberg Dordrecht London New York

Reprint of the 1st ed. C.I.M.E., Ed. Cremonese, Roma 1974
With kind permission of C.I.M.E.

Printed on acid-free paper

Springer.com

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

III Ciclo - Varenna dal 16 al 25 giugno 1974

EIGENVALUES OF NON-LINEAR PROBLEMS

Coordinatore: Prof. G. PRODI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

NONLINEAR EIGENVALUE PROBLEMS IN ORDERED BANACH SPACES

H. AMANN

Corso tenuto a Verenna dal 16 al 25 giugno 1974

NONLINEAR EIGENVALUE PROBLEMS

IN ORDERED BANACH SPACES

Herbert Amann

1. Introduction

In this paper we study nonlinear elliptic boundary value problems of the form

$$(1.1)\qquad \begin{aligned} Lx(t) &= \lambda\phi(t,x(t)) && \text{in } \Omega , \\ Bx(t) &= o && \text{on } \partial\Omega , \end{aligned}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$, L is a second order strongly uniformly elliptic real differential operator, B is an at most first order real boundary operator, and λ is a real number.

As for the pair (L,B) , we impose the following fundamental hypotheses:

(H1) *The pair* (L,B) *satisfies the strong maximum principle,* that is, for every function $x \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{aligned} Lx(t) &\geq o && \text{in } \Omega , \\ Bx(t) &\geq o && \text{on } \partial\Omega , \end{aligned}$$

it follows that $x(t) \geq o$ on $\overline{\Omega}$. Moreover, if $x \neq o$ then $x(t) > o$ for every $t \in \Omega$, and for every $t \in \partial\Omega$ such that $x(t) = o$, it follows that $\frac{\partial x}{\partial \nu}(t) < o$, where ν denotes the outer normal on $\partial\Omega$.

(H2) *The pair* (L,B) *satisfies Schauder type a priori estimates,* that is, there exists a constant $\gamma > o$ such that for every $x \in C^{2+\mu}(\overline{\Omega})$ satisfying the boundary conditions $Bx(t) = o$ on $\partial\Omega$,

$$\|x\|_{C^{2+\mu}(\overline{\Omega})} \leq \gamma \, \|Lx\|_{C^{\mu}(\overline{\Omega})},$$

where $\mu \in (o,1)$ if $N \geq 2$, and $\mu = o$ if $N = 1$.

It is well known that hypothesis (H2) is satisfied if the coefficients of L belong to $C^{\mu}(\overline{\Omega})$, $Bx = x|\partial\Omega$, and the homogeneous problem

$$Lx = o \;,\; Bx = o$$

has the trivial solution only. Clearly, this latter condition is satisfied if hypothesis (H1) holds. If B is the Dirichlet boundary operator, that is, $Bx = x|\partial\Omega$, then (H1) is satisfied if the coefficient of the non-differentiated term in L is nonnegative. For more general boundary conditions we refer to [1,2].

As for the nonlinearity ϕ, we impose the following hypothesis.

(H3) (i) $\phi \in C^2(\overline{\Omega} \times \mathbb{R}_+)$,

(ii) *there exist positive constants* γ_0 *and* δ *such that for* *every* $(t,\xi) \in \overline{\Omega} \times \mathbb{R}_+$, $\phi(t,\xi) \geq \gamma_0 + \delta\xi$,

(iii) *there exists a nonnegative constant* ω_0 *such that for every* $(t,\xi) \in \overline{\Omega} \times \mathbb{R}_+$, $D_2\phi(t,\xi) > -\omega_0$,
where D_2 denotes the partial derivative with respect to the second variable.

It should be remarked that for simplicity we restrict our considerations to nonlinear eigenvalue problems of the form (1.1). However, it is easily seen that the same methods apply to more general problems of the form

$$\begin{aligned} Lx(t) &= \psi(t,x(t),\lambda) && \text{in } \Omega \ , \\ Bx(t) &= \chi(t) && \text{on } \partial\Omega \ . \end{aligned}$$

It is an immediate consequence of the above hypotheses that for the solvability of (1.1) it is necessary that $\lambda \geq 0$. Moreover, if for some $\lambda > 0$, problem (1.1) has a solution x then $x \neq 0$ and $x(t) \geq 0$ for $t \in \overline{\Omega}$. In order to use this important information we transform the nonlinear elliptic eigenvalue problem (1.1) into an abstract fixed point equation depending on a real parameter in a suitable ordered Banach space.

Hypothesis (H2) implies that L has a continuous inverse L^{-1} mapping $C^\mu(\overline{\Omega})$ into $C^{2+\mu}(\overline{\Omega})$. It follows from (H1) that L^{-1} is a *positive linear operator*, that is, the images of nonnegative functions are nonnegative functions. Hence it seems natural to transform problem

(1.1) into a fixed point equation in the Banach space $C^{\mu}(\overline{\Omega})$ where this space is given the natural ordering. However, for technical reasons (the positive cone is not normal) we do not use the space $C^{\mu}(\overline{\Omega})$ but $C(\overline{\Omega})$. In fact, it will be necessary to consider a certain subspace of $C(\overline{\Omega})$ which is particularly well adapted to the differential equation.

Let T be a nonempty set and let $x : T \to \mathbb{R}$ be a function on T. Then we write $x \geq o$ if $x(t) \geq o$ for every $t \in T$, and $x > o$ if $x \geq o$ but $x \neq o$. In the latter case, x is said to be *positive*.

It is well known that for every $y \in C^{\mu}(\overline{\Omega})$ the linear boundary value problem (BVP)

$$Lx = y \quad \text{in} \quad \Omega ,$$
$$Bx = o \quad \text{on} \quad \partial\Omega ,$$

has a unique solution $x := Ky$ in $C^{2+\mu}(\overline{\Omega})$. By using the L_p-theory for elliptic BVPs and Sobolev type imbedding theorems it can be shown (e.g. [1]) that the linear operator K defined above has a continuous extension, denoted again by K, to a compact linear operator $K : C(\overline{\Omega}) \to C^1(\overline{\Omega})$, the *solution operator (for the pair* (L,B)).

It is a consequence of the strong maximum principle that the solution operator K is not only a positive linear operator but it maps every positive function into a function without zeros in Ω. The following lemma (which is proved in [1]) contains the precise statement of this fact in a form which can be generalized to abstract ordered Banach spaces.

(1.1) Lemma: The solution operator K *maps* $C(\overline{\Omega})$ *compactly into* $C^1(\overline{\Omega})$ *and* K *is* e_0*-positive, that is, there exists a positive function* $e_0 \in C(\overline{\Omega})$ *such that for every* $x \in C(\overline{\Omega})$ *with* $x > o$ *, there are positive numbers* α *and* β *such that*

(1.2) $$\alpha e_0 \leq Kx \leq \beta e_0 \quad .$$

For e_0 *we may take the unique solution of the linear* BVP

$$Lx = \mathbb{1} \quad \text{in} \quad \Omega \; ,$$
$$Bx = o \quad \text{on} \quad \partial\Omega \; ,$$

where $\mathbb{1}(t) = 1$ *for every* $t \in \Omega$.

In particular, Lemma (1.1) implies that K can be considered as a positive compact endomorphism of $C(\overline{\Omega})$ and there exists a positive number α such that $Ke_0 \geq \alpha e_0$. Consequently, the Krein-Rutman theorem [6] applies and it follows that the spectral radius $r(K)$ is positive and an eigenvalue of K having a positive eigenfunction x_0 . It is easily seen that $\lambda_0 := r(K)^{-1}$ is the smallest eigenvalue, the *principal eigenvalue*, of the linear eigenvalue problem

(1.3) $$Lx = \lambda x \quad \text{in} \quad \Omega \; ,$$
$$Bx = o \quad \text{on} \quad \partial\Omega \; ,$$

and x_0 is a positive eigenfunction of (1.3).

We denote by $C_+(\overline{\Omega})$ the *positive cone* in $C(\overline{\Omega})$, that is, $C_+(\overline{\Omega})$ consists of o and all positive continuous functions on $\overline{\Omega}$. Then we define a continuous and bounded map

$$F : C_+(\overline{\Omega}) \rightarrow C_+(\overline{\Omega})$$

by

$$F(x)(t) := \phi(t,x(t)) \quad \text{for} \quad x \in C_+(\overline{\Omega}) \text{ and } t \in \overline{\Omega} \ ,$$

that is, F is the Nemytskii operator for the function ϕ . Then it is easily verified that the nonlinear elliptic eigenvalue problem (1.1) is equivalent to the fixed point equation

$$x = \lambda KF(x)$$

on $C_+(\overline{\Omega})$.

It is an immediate consequence of hypothesis (H3(ii)) that for every $x \in C_+(\overline{\Omega})$,

$$KF(x) \geq \gamma_0 K\mathbb{1} + \delta Kx = \gamma_0 e_0 + \delta Kx \ .$$

This inequality can be used to prove the following nonexistence theorem.

(1.2) Theorem: Denote by λ_0 *the principal eigenvalue of the linear eigenvalue problem* (1.3). *Then the nonlinear elliptic* BVP (1.1) *has no solution if* $\lambda \geq \lambda_0/\delta$.

Proof: By the Krein-Rutman theorem the dual operator K has an eigenvector x^* to the eigenvalue $r(K) = \lambda_0^{-1}$ such that $(x^*,x) \geq o$ for every $x \in C_+(\overline{\Omega})$.

We claim that for every $x \in \dot{C}_+(\bar{\Omega}) := C_+(\bar{\Omega}) \setminus \{o\}$, $(x^*,x) > o$. Indeed, since K is e_0-positive, inequalities (1.2) imply that

(1.4) $$\alpha(x^*,e_0) \le (x^*,x) \le \beta(x^*,e_0) ,$$

where α and β are positive numbers depending on $x \in \dot{C}_+(\bar{\Omega})$. Suppose that $(x^*,e_0) = o$. Then the preceding inequalities imply that $(x^*,x) = o$ for every $x \in C_+(\bar{\Omega})$. Since every $x \in C(\bar{\Omega})$ can be represented in the form $x = y - z$ with $y,z \in C_+(\bar{\Omega})$, it follows that $(x^*,x) = o$ for every $x \in C(\bar{\Omega})$. Thus $x^* = o$, which contradicts the fact that x^* is an eigenvector. Consequently, $(x^*,e_0) > o$ and by (1.4), $(x^*,x) > o$ for every $x > o$.

Now suppose that for some $\lambda > o$ the eigenvalue problem (1.1) has a solution x , or, equivalently, that $x = \lambda KF(x)$. Since $F(x)$ is positive and K is e_0-positive, it follows that $x > o$. Moreover,

$$x = \lambda KF(x) \ge \lambda\gamma_0 e_0 + \lambda\delta Kx .$$

By applying to this inequality the functional x^* , we obtain

$$(x^*,x) \ge \lambda\gamma_0(x^*,e_0) + \lambda\delta(x^*,Kx) > \lambda\delta r(K)(x^*,x) .$$

Thus $\lambda\delta r(K) < 1$, that is, $\lambda < \lambda_0/\delta$. Q.E.D.

So far we have essentially only used the fact that the solution operator K is positive. It is to be expected that we can obtain better results if the full nonlinear map λKF is compatible with the ordering of the underlying space. By using the a priori bound of Theorem (1.2) we can achieve such a situation by means of the following simple device.

Let $\omega := \lambda_0 \omega_0/\delta$. Thenproblem (1.1) is obviously equivalent to the nonlinear BVP

$$\begin{aligned} (L + \omega)x(t) &= \lambda\phi(t,x(t)) + \omega x(t) \quad \text{in} \quad \Omega , \\ Bx(t) &= o \quad \text{on} \quad \partial\Omega . \end{aligned}$$

Denote by K_ω the solution operator for the pair $(L + \omega, B)$ and define

$$F_\omega : \mathbb{R}_+ \times C_+(\bar{\Omega}) \to C_+(\bar{\Omega})$$

by

$$F_\omega(\lambda,x) := \lambda F(x) + \omega x \quad .$$

Then the nonlinear elliptic eigenvalue problem (1.1) is equivalent to the fixed point equation

$$x = K_\omega F_\omega(\lambda,x) \; .$$

Since $(L + \omega, B)$ satisfies hypotheses (H1) and (H2) it follows again that K_ω is an e-positive compact linear operator from $C(\bar{\Omega})$ into $C^1(\bar{\Omega})$, where $e := K_\omega \mathbb{1}$. Since F_ω is continuous and bounded, the map

$$f : \mathbb{R}_+ \times C_+(\bar{\Omega}) \to C_+(\bar{\Omega})$$

defined by

$$f(\lambda,x) := K_\omega F_\omega(\lambda,x)$$

is completely continuous, that is, f is continuous and maps bounded sets into compact sets.

By Theorem (1.2) it suffices to consider the restriction of f to the set $[o,\lambda_0/\delta) \times C_+(\bar{\Omega})$. On this set the map f has the important property that it is *increasing*, that is, for every pair (λ,x) , $(\mu,y) \in [o,\lambda_0/\delta) \times C_+(\bar{\Omega})$ such that $\mu \le \lambda$ and $y \le x$,

(1.5) $$f(\mu,y) \le f(\lambda,x) \quad .$$

Indeed, for every such pair (λ,x) , (μ,y) , the positivity of K_ω implies the inequality

$$f(\lambda,x) - f(\mu,y) =$$

$$(\lambda - \mu)K_\omega F(x) + \mu K_\omega(F(x) - F(y) + \mu^{-1}\omega(x - y)) \geq \tag{1.6}$$

$$(\lambda - \mu)K_\omega F(x) + \mu K_\omega(F(x) - F(y) + \omega_0(x - y)) .$$

Hence (1.5) is a consequence of hypothesis (H3).

Now observe that, due to the e-positivity of K_ω , every solution x of the fixed point equation $x = f(\lambda,x)$ is comparable to the function e in the sense that it belongs to the subset

$$C_e(\overline{\Omega}) := \{x \in C(\overline{\Omega}) \mid \text{there exists a positive number } \alpha = \alpha(x) \text{ such that } -\alpha e \leq x \leq \alpha e\}$$

of $C(\overline{\Omega})$. More precisely, f maps all of $[0,\lambda_0/\delta) \times C_+(\overline{\Omega})$ into

$$C_{e,+}(\overline{\Omega}) := C_e(\overline{\Omega}) \cap C_+(\overline{\Omega}) .$$

Hence it suffices to consider the restriction f_e of f to $[0,\lambda_0/\delta) \times C_{e,+}(\overline{\Omega})$.

Clearly, $C_e(\overline{\Omega})$ is a vector subspace of $C(\overline{\Omega})$. It can be made into a Banach space by means of the e-*norm*,

$$\|x\|_e := \inf\{\alpha > 0 \mid -\alpha e \leq x \leq \alpha e\} .$$

The e-norm is stronger than the maximum norm and it is *monotone*, that is, $\|x\|_e \leq \|y\|_e$ whenever $0 \leq x \leq y$. The set $C_{e,+}(\overline{\Omega})$ is the positive cone of $C_e(\overline{\Omega})$, that is, it consists of 0 and all positive continuous functions belonging to $C_e(\overline{\Omega})$. With respect to the e-norm, $C_{e,+}(\overline{\Omega})$ is closed and has nonempty interior. In fact, x is an interior point of

$C_{e,+}(\overline{\Omega})$ if and only if there exist positive numbers α and β such that $\alpha e \le x \le \beta e$. A proof of these assertions can be found in [5] .

In the following, $C_e(\overline{\Omega})$ will always be given the e-norm. Hence $C_e(\overline{\Omega})$ is a Banach space which is continuously imbedded in $C(\overline{\Omega})$. If $e(t) > o$ for every $t \in \overline{\Omega}$, then $C_e(\overline{\Omega})$ is topologically isomorphic to $C(\overline{\Omega})$. In case of Dirichlet boundary conditions we have $e|\partial\Omega = o$ and it can be shown that $C_o^1(\overline{\Omega}) := \{x \in C^1(\overline{\Omega}) \mid x|\partial\Omega = o\}$ is continuously imbedded in $C_e(\overline{\Omega})$. These facts and the compactness of K_ω as a map from $C(\overline{\Omega})$ into $C^1(\overline{\Omega})$ imply that K_ω maps $C(\overline{\Omega})$ compactly into $C_e(\overline{\Omega})$ (comp. [1]) . Moreover, since K_ω is e-positive, it maps $\dot{C}_+(\overline{\Omega})$ into the interior of $C_{e,+}(\overline{\Omega})$, that is, K_ω is *strongly positive.*

Using these facts and the definition of f_e , it is easily seen that f_e has the following property:

(P1) The map

$$f_e : [o,\lambda_o/\delta) \times C_{e,+}(\overline{\Omega}) \to C_{e,+}(\overline{\Omega})$$

is completely continuous. The map $f_e(o,\cdot)$ has exactly one fixed point, namely $x = o$. There exists $\rho > o$ such that for every positive x with $\|x\|_e = \rho$ and every $\sigma \ge 1$, $f_e(o,x) \neq \sigma x$.

Moreover it follows from inequality (1.6) and the e-positivity of K_ω that the map f_e is *strongly increasing*, that is, for every pair of distinct points (λ,x) , $(\mu,y) \in [o,\lambda_o/\delta) \times C_{e,+}(\overline{\Omega})$ satisfying $\mu \le \lambda$ and $y \le x$,

$$f_e(\lambda,x) - f_e(\mu,y) \in \operatorname{int} C_{e,+}(\overline{\Omega}) \ .$$

The regularity hypothesis (H3(i)) implies the following further property of f_e, which, in turn, implies that f_e is strongly increasing.

(P2) The map f_e is twice continuously differentiable on $[0,\lambda_0/\delta) \times C_{e,+}(\overline{\Omega})$. For every $(\mu,y) \in (0,\lambda_0/\delta) \times C_{e,+}(\overline{\Omega})$, the derivative

$$f_e'(\lambda,x) = (D_1 f_e(\lambda,x), D_2 f_e(\lambda,x)) : \mathbb{R} \times C_e(\overline{\Omega}) \to C_e(\overline{\Omega})$$

is a strongly positive continuous linear operator.

These lengthy considerations show that we can use the information which is contained in the maximum principle and the existence and regularity theory for linear elliptic BVPs to transform the nonlinear elliptic eigenvalue problem (1.1) into an equivalent fixed point equation of the form

$$x = f_e(\lambda,x)$$

in a Banach space $C_e(\overline{\Omega})$ such that both, the Banach space and the map enjoy the relatively pleasant properties given above. In this form we can apply the theory of ordered Banach spaces and the methods of nonlinear functional analysis to obtain relatively precise information about the solvability of problem (1.1).

2. Nonlinear Eigenvalue Problems in Ordered Banach Spaces

Let E be a real Banach space. A subset $P \subset E$ is called a cone if P is closed, $P + P \subset P$, $\mathbb{R}_+ P \subset P$, and $P \cap (-P) = \{o\}$. Given a cone P in E, we define an ordering in E by setting $x \leq y$ iff $y - x \in P$. Then (E,P) is called an ordered Banach space (OBS) with positive cone P. The elements $x \in \dot{P} := P \setminus \{o\}$ are called positive and we write $x < y$ to mean that $y - x \in \dot{P}$. Throughout this paper we assume that

(E,P) *is an* OBS *whose positive cone has nonempty interior* $\overset{o}{P}$ *and whose norm is monotone, that is,* $\|x\| \leq \|y\|$ *whenever* $o \leq x \leq y$.

Observe that the OBSs $(\mathbb{R}, \mathbb{R}_+)$ and $(\mathbb{R} \times E, \mathbb{R}_+ \times P)$ enjoy the above properties also, where the latter space is endowed with one of the usual norms, e.g. $\|(\lambda,x)\| = |\lambda| + \|x\|$.

We consider an equation of the form

$$x = f(\lambda,x) \quad , \tag{2.1}$$

where the map f satisfies the following assumptions:

$f : \mathbb{R}_+ \times P \to P$ *and is completely continuous.*

The map $f(o,\cdot) : P \to P$ *has exactly one fixed point, namely* $x = o$.

There exists $\rho > o$ *such that for every* $x \in P$ *with* $\|x\| = \rho$

every $\sigma \geq 1$, $f(o,x) \neq \sigma x$. *There exists* $\overline{\lambda} > o$ *such that* $f(\lambda,\cdot)$ *has no fixed point for* $\lambda \geq \overline{\lambda}$.

The map f *is twice continuously differentiable on* $[o,\overline{\lambda}) \in P$ *and for every* $(\lambda,x) \in (o,\overline{\lambda}) \times P$, *the derivative*

$f'(\lambda,x) = (D_1 f(\lambda,x) , D_2 f(\lambda,x)) : \mathbb{R} \times E \to E$

is a strongly positive linear operator, that is,

$f'(\lambda,x)((\mathbb{R}_+ \times P)^{\dot{}}) \subset \overset{o}{P}$.

We denote by Σ the solution set of equation (2.1), that is,

$$\Sigma := \{(\lambda,x) \in \mathbb{R}_+ \times P \mid x = f(\lambda,x)\} .$$

It follows from the results of Dancer [4] that the solution set is locally compact and contains an unbounded component emanating from (o,o) .

In the following we give more detailed information about the structure of Σ . For this purpose we denote by Λ the projection of Σ into $\mathbb{R}_+$, that is,

$$\Lambda = \{\lambda \in \mathbb{R} \mid f(\lambda,\cdot) \text{ has a fixed point}\} .$$

Since the derivative f' is strongly positive on $(o,\overline{\lambda}) \times P$ it follows that f is strongly increasing on this set, that is, for every pair of distinct points (λ,x) , $(\mu,y) \in (o,\overline{\lambda}) \times P$ such that $(\lambda,x) < (\mu,y)$, it follows that $f(\mu,y) - f(\lambda,x) \in \overset{o}{P}$. By using this fact, it is easy to prove the following theorem.

__*(2.1) Theorem:*__ Λ *is a nontrivial interval containing* o . *For every* $\lambda \in \Lambda$, *there exists a minimal fixed point* $\overline{x}(\lambda)$ *of* $f(\lambda,\cdot)$.

The map $\overline{x}(\cdot) : \Lambda \to P$ *is strongly increasing and left continuous.*

Clearly, $\overline{x}(\lambda)$ is a minimal fixed point iff every fixed point x_λ of $f(\lambda,\cdot)$ satisfies $x_\lambda \geq \overline{x}(\lambda)$. In particular, the minimal fixed point is unique. In the proof of Theorem (2.1) it is shown that $\overline{x}(\lambda)$ can be computed iteratively, namely $\overline{x}(\lambda) = \lim_{k \to \infty} f^k(\lambda,o)$, that is, $\overline{x}(\lambda)$ is the limit of the sequence (x_k) , where $x_0 = o$ and $x_{k+1} = f(\lambda,x_k)$.

We set $\lambda^* := \sup \Lambda$. Then $o < \lambda^* \leq \overline{\lambda}$. Next we impose a further hypothesis, namely the existence of an a priori bound.

(H) *There exist* $\mu \in (o,\lambda^*)$ *and* $\rho > \|\overline{x}(\mu)\|$ *such that there is no* $(\lambda,x) \in \Sigma$ *with* $\lambda \geq \mu$ *and* $\|x\| = \rho$.

It is an easy consequence of hypothesis (H) that Λ is closed, that is, $\lambda^* \in \Lambda$. Consequently, $\lambda^* < \overline{\lambda}$.

Now we claim that for every $\lambda \in [\mu,\lambda^*)$, the map $f(\lambda,\cdot)$ has at least two distinct fixed points. By Theorem (2.1) we know already that for every such λ , there exists a minimal fixed point $\overline{x}(\lambda)$. Suppose now that λ is a point of discontinuity of $\overline{x}(\cdot)$. Then it is easy to show that $x_\lambda := \lim_{\sigma \downarrow \lambda} \overline{x}(\sigma)$ exists and is a fixed point of $f(\lambda,\cdot)$ with $x_\lambda > \overline{x}(\lambda)$. Hence it suffices to consider the case where λ is a point of continuity and $\overline{x}(\lambda)$ is an isolated fixed point of $f(\lambda,\cdot)$. In this case we show that the Leray-Schauder fixed point index of $\overline{x}(\lambda)$ is equal to + 1 .

It is an important consequence of the fact that $\overline{x}(\lambda)$ is the minimal fixed point of $f(\lambda,\cdot)$ that

$$r(D_2 f(\lambda,\overline{x}(\lambda))) \leq 1 \quad ,$$

where r denotes the spectral radius. Suppose now that $r(D_2 f(\lambda,\overline{x}(\lambda))) < 1$. Then it is an easy consequence of the standard Leray-Schauder degree theory that for every sufficiently small $\rho > o$

(2.2) $$d(id - f(\lambda,\cdot) , \overline{x}(\lambda) + B_\rho , o) = 1 \quad ,$$

where B_ρ denotes the open ball in E about o and radius ρ .

It remains to consider the more difficult case where $\overline{x}(\lambda)$ is an isolated fixed point of $f(\lambda,\cdot)$ with

$$r(D_2 f(\lambda,\overline{x}(\lambda))) = 1 \quad .$$

In this case, 1 is a simple eigenvalue of $D_2 f(\lambda,\overline{x}(\lambda))$. By means of this knowledge and several applications of the implicit function theorem it is possible to show that in a neighborhood of the point $(\lambda,\overline{x}(\lambda))$, the solution set Σ consists of a smooth curve $(\lambda(\sigma),x(\sigma))$ where $\sigma \in (-\varepsilon,\varepsilon)$ and the map $\sigma \to x(\sigma)$ is strongly increasing. Using this local representation of Σ and the homotopy invariance of the Leray-Schauder degree, it can finally be shown that also in this case relation (2.2) is true.

Now we show that for a sufficiently large open subset U of $\overset{o}{P}$ which contains the minimal fixed point of $f(\lambda,\cdot)$, the Leray-Schauder degree vanishes. Then it follows from (2.2) and the additivity property of the Leray-Schauder degree, that there must be at least one more fixed point in U .

Since $\overline{x}(\cdot)$ is strongly increasing, there exists $x_0 \in \overset{o}{P}$ such that $\overline{x}(\mu) - x_0 \in \overset{o}{P}$. Hence hypothesis (H) and Theorem (2.1) imply that for every $\lambda \geq \mu$, no fixed point of $f(\lambda,\cdot)$ is contained on the boundary of the bounded open subset

$$U := (x_0 + \overset{o}{P}) \cap B_\rho$$

of E . Consequently, the Leray-Schauder degree $d(id - f(\lambda,\cdot),U,o)$ is well defined and, due to the homotopy invariance,

$$d(id - f(\lambda,\cdot),U,o) = d(id - f(\overline{\lambda},\cdot),U,o)$$

for every $\lambda \geq \mu$. But since $f(\overline{\lambda},\cdot)$ has no fixed points at all, it follows that

$$d(id - f(\lambda,\cdot),U,o) = o$$

for every $\lambda \geq \mu$.

By this way we obtain the following theorem whose detailed proof is given in [2] .

(2.2) Theorem: Let hypothesis (H) *be satisfied. Then there exists* $\lambda^* > o$ *such that problem* (2.1) *has at least one solution for every* $\lambda \in [o,\lambda^*]$ *, no solution for* $\lambda > \lambda^*$ *, and at least two distinct solutions for* $\mu \leq \lambda < \lambda^*$.

3. Applications to Elliptic Boundary Value Problems

As has been shown in the introduction, the results of the preceding paragraph apply to the elliptic eigenvalue problem (1.1) provided the additional hypothesis (H) can be verified. This means that a priori bounds for the solutions of problem (1.1) have to be established.

In the following we exhibit two cases for which the necessary a priori bounds can be established. Namely, the case of two point boundary value problems of ordinary differential equations and the case of asymptotically linear nonlinearities.

In the remainder of this section we suppose that hypotheses (H1) - (H3) are satisfied.

(3.1) Theorem: Suppose that $N = 1$ *, that is,* L *is a second order ordinary differential operator. Moreover suppose that*

$$\lim_{\xi \to \infty} \frac{\phi(t,\xi)}{\xi} = \infty$$

uniformly in $t \in \overline{\Omega}$ *. Then there exists a positive number* λ^* *such that problem* (1.1) *has no solution for* $\lambda > \lambda^*$*, at least one solution for* $\lambda = \lambda^*$*, and at least two positive solutions for* $0 < \lambda < \lambda^*$.

A detailed proof of this theorem is given in [3] .

The next theorem applies to elliptic BVPs in any dimensions. But the nonlinearity is supposed to be asymptotically linear.

(3.2) Theorem: *Suppose that there exists* $\phi_\infty \in C^\mu(\overline{\Omega})$ *such that*

$$\phi_\infty(t) = \lim_{\xi \to \infty} D_2\phi(t,\xi)$$

uniformly in $t \in \overline{\Omega}$. *In addition suppose that there exists* $y \in \dot{C}_+(\overline{\Omega})$ *and a constant* $\rho > 0$ *such that for all* $t \in \Omega$ *and all* $\xi \geq \rho$,

$$\phi(t,\xi) - D_2\phi(t,\xi) \cdot \xi \leq -y(t) .$$

Denote by λ_∞ *the principal eigenvalue of the linear eigenvalue problem*

$$Lx = \lambda\phi_\infty x \qquad in \quad \Omega \quad ,$$

$$BBx = 0 \qquad on \quad \partial\Omega \quad .$$

Then there exists a positive number λ^* *such that problem* (1.1) *has no solution for* $\lambda > \lambda^*$ *and at least one solution for* $0 \leq \lambda \leq \lambda^*$. *Furthermore,* $0 < \lambda_\infty < \lambda^*$ *and for every* $\lambda \in (\lambda_\infty, \lambda^*)$, *problem* (1.1) *has at least two distinct solutions.*

For a proof of this theorem as well as for further details and bibliographic references we refer to [2] .

H. Amann

Bibliography

[1] H. AMANN : On the number of solutions of nonlinear equations in ordered Banach spaces.
J. Functional Anal. 11 (1972), 346-384.

[2] ————: Multiple positive fixed points of asymptotically linear maps.
J. Functional Anal., to appear

[3] ————: On the number of solutions of asyptotically superlinear two point boundary value problems.
Arch. Rat. Mech. Anal., to appear.

[4] E.N. DANCER: Global solution branches for positive mappings.
Arch. Rat. Mech. Anal. 52 (1973), 181-192.

[5] M.A. KRASNOSEL'SKII: Positive Solutions of Operator Equations.
Groningen: Noordhoff 1964.

[6] M.G. KREIN and M.A. RUTMAN: Linear operators leaving invariant a cone in a Banach space.
Amer.Math. Soc. Translations, Ser.1,1o,1-128 (1962).

Mathematisches Institut der
Ruhr-Universität
D 463o Bochum, Germany