A. Figà Talamanca (Ed.)

Harmonic Analysis and Group Representation

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NILPOTENT GROUPS AND ABELIAN VARIETIES

L. AUSLANDER AND R. TOLIMIERI

Lectures on
NILPOTENT GROUPS AND ABELIAN VARIETIES
by
L. Auslander and R. Tolimieri

Introduction

A. A. Albert, in an immense burst of creative energy succeeded in solving the "Riemann matrix problem." Although this is one of the great mathematical achievements of our century, there are few systematic accounts of Albert's work. Perhaps, C. L. Siegel's account [6] comes the closest to providing us with a view of this marvelous achievement. Albert's and Siegel's treatment are difficult because their arguments are based on matrix calculations. Because a coordinate system has been chosen, there is a hidden identification of a vector space with its dual and matrices play the role of both linear transformations and bilinear forms.

In these notes, we will present a way of using nilpotent groups to formulate the ideas of Abelian varieties and present part of the existence theorems contained in Albert's work. A full treatment of the existence part of Albert's work will appear in [4]. Our approach rests on nilpotent algebraic groups. This enables us to present a matrix-free treatment of the Riemann matrix problem. We hope this approach will reawaken admiration for, and interest in, Albert's achievement.

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1. Associative Algebras and Nilpotent Algebraic Groups

In these notes the word field will denote either the reals, $\mathbb{R}$, the complex, $\mathbb{C}$, or an algebraic number field, $\mathscr{k}$, containing the rationals, $\mathbb{Q}$, and we let $[\mathscr{k},\mathbb{Q}] = h < \infty$. Further, all algebras and vector spaces are finite dimensional and all associative algebras have an identity.

Let $\mathscr{N}$ be an associative algebras over $\mathscr{k}$ of the form $\mathscr{k} \cdot 1 \oplus \mathscr{R}$, where 1 is the identity of $\mathscr{N}$, and $\mathscr{R}$ is the radical of $\mathscr{N}$. Let $N(\mathscr{N})$ be the subset of the group of units of $\mathscr{N}$ of the form

$$N(\mathscr{N}) = \{1 + n \mid n \in \mathscr{R}\} .$$

Then $N(\mathscr{N})$ is a subgroup of the group of units, because

$$(1 + n_2)(1 + n_2) = 1 + n_1 + n_2 + n_1 n_2$$

and because $\mathscr{R}$ is an ideal if $n_1, n_2 \in \mathscr{R}$, then $n_1 + n_w + n_1 n_2 \in \mathscr{R}$. Since $\mathscr{R}$ is nilpotent, there exists k such that $\mathscr{R}^{k+1} = \{0\}$. By the binomial theorem,

$$(1 + n)^{-1} = 1 - n + \dots + (-1)^k n^k .$$

Let $\mathscr{I} \subset \mathscr{R}$ be an ideal in $\mathscr{R}$. Let

$$G(\mathscr{I}) = \{(1 + n) \mid n \in \mathscr{I}\} .$$

Then $G(\mathscr{I})$ is easily seen to be a normal subgroup of $N(\mathscr{N})$ and $N(\mathscr{N})/G(\mathscr{I})$ is isomorphic to $N(\mathscr{N}/\mathscr{I})$.

Let the dimension of $N(\mathscr{N})$ as a $\mathscr{k}$-vector space be m. For $1+n \in N(\mathscr{N})$, and $g \in \mathscr{N}$, the mapping

$$\rho(1 + n)(g) = (1 + n)g$$

is a representation of $N(\mathscr{N})$ in $GL(m,\mathscr{k})$. Further, there is a basis of $\mathscr{N}$ such that $\rho(N(\mathscr{N})) \subset U(m,\mathscr{k})$, where

$$U(m,\mathscr{k}) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

and $\rho(N(\mathscr{N}))$ is the set of zeros of a set of linear equations over $\mathscr{k}$. We call $N(\mathscr{N})$ the $\mathscr{k}$-algebraic group of N. It is easily verified that $N(\mathscr{N})$ is a nilpotent group.

The above has the following generalization.

Definition: Let G be a nilpotent group. We call G a k-nilpotent algebraic group if there exists an isomorphism $\rho: G \longrightarrow U(m,k)$ such that $\rho(G)$ is a k-algebraic variety in $U(m,k)$.

A nilpotent group G is called 2-step nilpotent if [G,G] is central. If G is a 2-step k-nilpotent algebraic group it is easily verified that G satisfies an exact sequence:

$$0 \to V_1 \to G \to V_2 \to 0,$$

where V_1 and V_2 are k-vector spaces such that $V_1 \supset [G,G]$ and V_1 is in the center of G. With these general definitions out of the way, we can discuss the special cases with which we will be concerned in these notes. Let V be a k-vector space and let $\Lambda(V)$ denote the exterior algebra over V. Then

$$\Lambda(V) = k \cdot 1 \oplus \mathscr{R}$$

where $\mathscr{R} = \sum_{i>0} \Lambda^i(V)$ is the radical of $\Lambda(V)$.. Hence, we may form the k-nilpotent algebraic group $N(\Lambda(V))$. It is clear that $\mathscr{I} = \sum_{i>2} \Lambda^i(V)$ is an ideal in $\Lambda(V)$. Hence we may form

$$\mathscr{F}_2(V) = N(\Lambda(V))/G(\mathscr{I}) = N(\Lambda(V)/\mathscr{I}).$$

Since $\mathscr{F}_2(V)$ is very important in the rest of this paper, we will present another more explicit description or "presentation" of $\mathscr{F}_2(V)$. As a set

$$\mathscr{F}_2(V) = V \times V \wedge V$$

and the group law of composition is given by

$$(v_1,w_1)(v_2,w_2) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2)$$

where $v_2 \epsilon V$ and $w_2 \epsilon V \wedge V$ for $\alpha = 1,2$.

$\mathscr{F}_2(V)$ is a 2-step k-nilpotent algebraic group with center $(0,w)$, $w \epsilon V \wedge V$, and it is called the free 2-step k-nilpotent group over V. The reason for the name, "free," is the following: If G is a 2-step k-nilpotent algebraic group and

$$f: \mathscr{F}_2(V)/[\mathscr{F}_2(V), \mathscr{F}_2(V)] \longrightarrow G/[G,G]$$

is k-linear, then there exists a homomorphism

$$F: \mathscr{F}_2(V) \longrightarrow G$$

such that the kernel of F is a $\mathscr{k}$-algebraic group and the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{F}_2(V) & \xrightarrow{F} & G \\ \downarrow & & \downarrow \\ \mathscr{F}_2(V)/[\mathscr{F}_2(V),\mathscr{F}_2(V)] & \xrightarrow{f} & G/[G,G] \end{array}$$

The nilpotent algebraic groups $N(\Lambda(V))$ and $\mathscr{F}_2(V)$ exhibit a property that will have enormous implications in our later work. We observe that the representations of $N(\Lambda(V))$ or $\mathscr{F}_2(V)$ arising from the associative algebra structure can be defined by linear equations whose coefficients are in $\mathbb{Q}$ This will enable us to consider $N(\Lambda(V))$ and $\mathscr{F}_2(V)$ as $\mathbb{Q}$-nilpotent algebraic groups. We will now discuss how this can be done.

Let G be a $\mathscr{k}$-nilpotent algebraic group and assume that a set of equations defining G can be chosen to have coefficients in $K \subset \mathscr{k}$. We will then say that G is defined over K. Now let V be an m-dimensional $\mathscr{k}$-vector space. If $[\mathscr{k}:K] = h$, then we may consider V as an mh dimensional K vector space that we will denote by $V(K)$. Clearly, $\mathscr{k}$ linear transformation of V gives rise to a K linear transformation of $V(K)$. Thus we have an isomorphism

$$r(K):GL(m,\mathscr{k}) \longrightarrow GL(mh,K) .$$

We will call $r(K)$ the isomorphism of reducing the field from $\mathscr{k}$ to K . It is easily seen that if G is a $\mathscr{k}$-algebraic group defined over K, then $r(K)(G)$ will be a K-algebraic group. We will call $r(K)(G)$ the K-algebraic group obtained by reducing the field of G.

Again, let G be a $\mathscr{k}$-algebraic group defined over K, $K \subset \mathscr{k}$. Consider $G(K) \subset G$ consisting of those points in $GL(m,\mathscr{k})$, all of whose coefficients are in K. Then $G(K)$ will be a K-algebraic group in $GL(m,K)$. If all the $\mathscr{k}$ points of $G(K) = G$, we will call $G(K)$ a K-form of the $\mathscr{k}$-algebraic group G. It should be remarked, that G may have non-isomorphic K-forms.

An example may help the reader understand all this better. Let $\mathscr{k}$ be a totally real algebraic number field over $\mathbb{Q}$ and let $[\mathscr{k}:\mathbb{Q}] = h$. Consider the $\mathscr{k}$-algebraic subgroup G of $GL(2,k)$ defined by

$$G = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x \in \mathscr{k}$$

A set of defining equations for G are given by $x_{11} = x_{22} = 1$ and $x_{21} = 0$ where

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = GL(2,k)$$

Clearly, G may also be considered as the $\mathscr{k}$-points of the $\mathbb{Q}$ algebraic group

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \subset GL(2,\mathbb{Q}) \quad x \in \mathbb{Q}$$

We will now give an explicit map for $r(\mathbb{Q})$. Let r denote the regular representation of $\mathscr{k}$ over $\mathbb{Q}$. Then

$$r(\mathbb{Q})\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} r(x_{11}) & r(x_{12}) \\ r(x_{21}) & r(x_{22}) \end{pmatrix}$$

where we view the right hand matrix in $GL(2h,\mathbb{Q})$. Thus $r(\mathbb{Q})(G) = G(\mathbb{Q}) \subset GL(2h,\mathbb{Q})$ is a $\mathbb{Q}$-algebraic group. Let $G(\mathbb{Q})_{\mathbb{R}}$ denote the group of $\mathbb{R}$-points of $G(\mathbb{Q})$. Then $G(\mathbb{Q})_{\mathbb{R}} \subset GL(2h,\mathbb{R})$. Since $\mathscr{k}$ is totally real, there exists $A \in GL(h,\mathbb{R})$ such that

$$A^{-1}r(k)A = \begin{pmatrix} \chi_1(k) & & 0 \\ & \ddots & \\ 0 & & \chi_h(k) \end{pmatrix} = D(k) k \in \mathscr{k}$$

and $\chi_i : k \longrightarrow \chi_i(k)$, $i = 1,\ldots,h$, is an isomorphism of $\mathscr{k}$ into $\mathbb{R}$. Indeed, $\chi_1,\ldots,\chi_n$ are distinct isomorphisms. Now

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} G(\mathbb{Q})_{\mathbb{R}} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} I & D \\ 0 & I \end{pmatrix}$$

where D is as above.

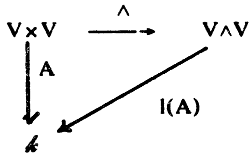
Now, $N(\Lambda(V))$ and $\mathscr{F}_2(V)$ are easily seen to be defined over $\mathbb{Q}$ and so both may be considered - by reducing the field - as $\mathbb{Q}$- algebraic groups. Hence, we have the identity map $V(\mathbb{Q}) \longrightarrow V$ lifts to a morphism

$$\mathscr{F}_2(V(\mathbb{Q})) \longrightarrow \mathscr{F}_2(V).$$

There are certain homomorphisms of $\mathcal{F}_2(V)$ that will play an essential role in our theory. We will now establish a language with which to carry out this discussion. We begin by listing some standard notation that we will follow. If V and W are k-vector spaces, we use $\mathrm{Hom}(V,W)$ to denote the k-vector space of k-linear maps and $V^* = \mathrm{Hom}(V,k)$, the dual vector space. For $T \in \mathrm{Hom}(V,W)$, we have $T^* \in \mathrm{Hom}(W^*,V^*)$ and we will identify V^{**} with V.

Let $\mathrm{Bil}(V)$ denote the vector space of bilinear forms on $V \times V$. For $B \in \mathrm{Bil}(V)$, define $L(B) \in \mathrm{Hom}(V,V^*)$ by $(L(B)(u))(v) = B(u,v)$, $u,v \in V$. Since $L(B) \in \mathrm{Hom}(V,V^*)$, we have $L(B)^* \in \mathrm{Hom}(V,V^*)$. Clearly, B is alternating if and only if $L(B)^* = -L(B)$, and B is symmetric if and only if $L(B)^* = L(B)$. The set of alternating forms will be denoted by $\mathrm{Alt}(V)$, $\mathrm{Sym}(V)$ will denote the set of symmetric forms, and $\mathrm{Bil}(V) = \mathrm{Alt}(V) \oplus \mathrm{Sym}(V)$. If $L(B)$ is nonsingular, we say that B is non-singular and the space of non-singular bilinear forms will be denoted by $\mathrm{Bil}^\times(V)$. Analogously, we will use the notation $\mathrm{Alt}^\times(V) = \mathrm{Alt}(V) \cap \mathrm{Bil}^\times(V)$ and $\mathrm{Sym}^\times(V) = \mathrm{Sym}(V) \cap \mathrm{Bil}^\times(V)$.

Let $\mathcal{F}_2(V)$ denote the free 2-step k-nilpotent group over V. The dual space to $V \wedge V$ is $V^* \wedge V^*$, and we have the commutative diagram



where $A \in \mathrm{Alt}(V)$ and $l(A) \in V^* \wedge V^*$; this enables us to identify $\mathrm{Alt}(V)$ and $V^* \wedge V^*$.

Now, for $A \in \mathrm{Alt}(V)$, we may define a group structure $N(A)$ on the set $V \times k$ whose law of multiplication is

$$(v_1,k_1)(v_2,k_2) = (v_1 + v_2, k_1 + k_2 + A(v_1,v_2))$$

where $v_1,v_2 \in V$ and $k_1,k_2 \in k$. Then $N(A)$ has (o,k), $k \in k$ in its center and $N(A)$ modulo its center is Abelian. Hence, $N(A)$ is a 2-step k-nilpotent algebraic group. Define the surjection

$$P : \mathcal{F}_2(V) \longrightarrow N(A)$$

by $P(v,k) = (v, l(A)w)$, $(v,w) \in \mathcal{F}_2(V)$. If $i : V \longrightarrow V$ is the identity mapping, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}_2(V) & \xrightarrow{P} & N(A) \\ \downarrow & & \downarrow \\ \mathcal{F}_2(V)/[\mathcal{F}_2(V),\mathcal{F}_2(V)] & \longrightarrow & N(A)/[N(A),N(A)] \end{array}$$

We will call such morphisms of $\mathcal{F}_2(V)$ *polarizations and denote the set of polarizations by* P(V). Clearly, we may identify P(V) with Alt(V) as above. If $A \in \mathrm{Alt}^{\times}(V)$, all N(A) are isomorphic and we will call N(A) a $\mathfrak{k}$-Heisenberg group. The corresponding polarizations will be denoted by $P^{\times}(V)$. If dim V = 2m, we will sometimes use $N_{2m+1}(\mathfrak{k})$ to denote N(A) and call $N_{2m+1}(\mathfrak{k})$ the 2m+1 $\mathfrak{k}$-Heisenberg group. Fixing an isomorphism of the center $\mathfrak{z}$ of $N_{2m+1}(\mathfrak{k})$ with $\mathfrak{k}$, as when we present $N_{2m+1}(\mathfrak{k})$ as N(A), will be called an *orientation* of $N_{2m+1}(\mathfrak{k})$.

The presentation N(A) of the Heisenberg $N_{2m+1}(\mathfrak{k})$ has the additional property of determining an isomorphism which we will denote by $A:V \longrightarrow V^*$ or, if P corresponds to A, by $P:V \longrightarrow V^*$. This follows from the fact that A is non-degenerate.

2. The Jacobi Variety of a Riemann Surface and Abelian Varieties

In this lecture we will need two special examples of a general phenomena; accordingly, we will begin with the general case and then specialize to the examples of interest to us.

Let M be a compact manifold and let $H^*(M,\mathbb{R})$ and $H^*(M,\mathbb{Z})$ be the cohomology rings of M with real and integer coefficients, respectively. If $\mathcal{R} = \sum_{i>0} H^i(M,\mathbb{R})$, then $\mathcal{R}$ is the radical of $H^*(M,\mathbb{R})$ and $H^*(M,\mathbb{R}) = \mathbb{R} \oplus \mathcal{R}$. Hence, we may form the nilpotent algebraic group $N(H^*(M,\mathbb{R}))$, which we will henceforth denote by N(M). Now the Lie algebra of N(M) is the Lie algebra associated with $\mathcal{R}$ by

$$[x,y] = xy - yx \qquad x,y \in \mathcal{R}.$$

Since, for $x \in H^i(M,\mathbb{R})$ and $y \in H^j(M,\mathbb{R})$, we have

$$xy = (-1)^{ij}\, yx \in H^{i+j}(M,\mathbb{R}) .$$

It follows that, if $x \in H^{2i}(M,\mathbb{R})$, then

$$[x,y] = xy - yx = 0 ,$$

and so $\sum H^{2i}(M,\mathbb{R}) \subset \mathcal{R}$ is in the center of $\mathcal{R}$ as a Lie algebra. Further, if $x,y \in \sum H^{2i+1}(M,\mathbb{R})$. Then $[x,y] \in \sum H^{2i}(M,\mathbb{R})$. Thus $\mathcal{R}$ is a 2-step nilpotent Lie algebra and so N(M) is a 2-step nilpotent Lie group.

By the standard theory of cohomology rings, there is a natural injection

$$i: H^*(M,\mathbb{Z}) \longrightarrow H^*(M,\mathbb{R})$$

such that $i(H^*(M,\mathbb{Z}))$ is a lattice in the vector space $H^*(M,\mathbb{R})$. Let $\mathcal{R}(\mathbb{Z}) = i(H^*(M,\mathbb{Z})) \cap \mathcal{R}$. Then we may argue as before and obtain that

$$\Gamma(M) = \{1 + n \mid n \in \mathcal{R}(\mathbb{Z})\}$$

is a subgroup of N(M). It is then easily verified that $\Gamma(M)\backslash N(M)$ is a compact manifold, called a nilmanifold. Hence, we have functorally assigned to every compact manifold M, the compact nilmanifold $\Gamma(M)\backslash N(M)$. By [1], there exists a unique $\mathbb{Q}$-nilpotent algebraic group $N_{\mathbb{Q}}(M)$ such that $\Gamma(M) \subset M_{\mathbb{Q}}(M) \subset N(M)$. We will call $N_{\mathbb{Q}}(M)$ the topological rational form of N(M). (It may happen that N(M) has other rational forms not isomorphic to $N_{\mathbb{Q}}(M)$).

The groups N(M) and P(M) constructed above have an additional structure that we will now discuss.

As a set N(M) = X × Y , where

$$X = \{1+n \mid n \in \sum H^i(M,\mathbb{R}),\ i \text{ odd}\}$$

$$Y = \{1+n \mid n \in \sum H^i(M,\mathbb{R}),\ i \text{ even},\ i > 0\}$$

where X and Y are vector spaces. If (x,y) ∈ X × Y, then the multiplication in N(M) is given by

$$(x_1,y_1)(x_2,y_2) = (x_1 + x_2,\ y_1 + y_2 + B(x_1,x_2))$$

where B : X × X ⟶ Y is skew symmetric. Such a presentation of a 2-step k-algebraic group will be called a grading. Notice that the presentation of $\mathcal{F}_2$(V) as V × V∧V in Section 1 was a graded presentation and N(A) was a graded presentation of the Heisenberg group.

The main purpose for introducing the graded structure of 2-step k-nilpotent algebraic groups is the following: If N = X × Y is a graded 2-step k-nilpotent algebraic group and if α : V ⟶ X is a morphism, then α has a *unique* extension to a morphism $\alpha^*:\mathcal{F}_2(V) \longrightarrow N$ that preserves gradings. This is because the composite mapping

$$V \times V \xrightarrow{\alpha \times \alpha} X \times X \xrightarrow{B} Y$$

is an alternating bilinear mapping on V × V and so we have a *unique* linear mapping ℓ(B):V∧V ⟶ Y that completes the commutative diagram

$$\begin{array}{ccc} V \times V & \xrightarrow{\wedge} & V \wedge V \\ {\scriptstyle B \circ \alpha \times \alpha} \downarrow & \swarrow {\scriptstyle \ell(B)} & \\ Y & & \end{array}$$

It follows that if A is any morphism of V then A determines a unique graded morphism of $\mathcal{F}_2$(V).

For the rest of this paper, we will restrict ourselves to graded nilpotent algebraic groups and all morphisms will be grading preserving morphism. *Henceforth, we will drop the word graded, but it will be what assures the uniqueness of various morphisms that occur in the discussion.*

Let M be a complex manifold. Then as in [9], the complex structure on M determines an automorphism J(M) of $H^*(M,\mathbb{R})$. Further, if

$$f:M_1 \longrightarrow M_2$$

is a complex analytic mapping, then

$$f^*J(M_2) = J(M_1)f^* .$$

It is clear that the complex structure determines an automorphism of N(M), which we will also denote by J(M). It is important to note that J(M) may *not* induce an automorphism of Γ(M) or even of $N_{\mathbb{Q}}(M)$.

To illustrate this, let us see how all this works for the m complex dimensional torus. Let W be an m dimensional complex vector space and let L be a discrete subgroup of W such that W/L is compact. We will begin by discussing another way of looking at W. Clearly, W is also a real vector space $W(\mathbb{R})$ of real dimension 2m. Let $e_1,\ldots,e_m$ be a basis of W. Then $e_1, ie_1, \ldots, e_m, ie_m$, $i = \sqrt{-1}$, is a basis of $W(\mathbb{R})$. For w ∈ W, the mapping $J:W \longrightarrow iW$ defines an automorphism of $W(\mathbb{R})$ which in terms of the above basis is given by

$$J = \begin{pmatrix} J_0 & & 0 \\ & \ddots & \\ 0 & & J_0 \end{pmatrix} = m\, J_0 \text{ where } J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that J has the property that $J^2 = -I$, where I is the identity mapping.

Let A be a real linear transformation of $W(\mathbb{R})$. When does A induce a *complex* linear transformation of W? We will now verify that the answer is when

$$JA = AJ .$$

By a straightforward computation, one verifies that

$$J_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} J_0$$

if and only if a = d, b = −c. But since the regular representation r of $\mathbb{C}$ over $\mathbb{R}$ is given by

$$r{:}a + ib \longrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

we have that our assertion is true for m = 1. Relative to the basis $e_1,\ldots,e_m$, let

$$C = (C_{\alpha\beta}) \in Hom(W,W) \qquad \alpha,\beta = 1,\ldots,m .$$

Then relative to the basis $e_1, ie_1,\ldots,e_m, ie_m$ of W($\mathbb{R}$), C is given by

$$C = (r(C_{\alpha\beta})) .$$

It then follows easily that JC = CJ.

Now assume that JA = AJ and write A as an m×m matrix whose entries are 2×2 matrices

$$A = (A_{\alpha\beta}) \qquad \alpha,\beta = 1,\ldots,m .$$

By a direct computation, we have that JA = AJ implies that

$$J_0 A_{\alpha\beta} = A_{\alpha\beta} J_0 \qquad \text{all } \alpha,\beta .$$

Hence each $A_{\alpha\beta} = r(C_{\alpha\beta})$, $C_{\alpha\beta} \in \mathbb{C}$ and we have that A gives a complex linear transformation of W.

Now let V be a 2m dimensional real vector space and let J be an $\mathbb{R}$-linear transformation such that $J^2 = -I$. From the pair (V,J), we will construct a complex m-dimensional vector space W such that W($\mathbb{R}$) = V and the automorphism J:w ⟶ iw is the mapping J.

Let $e_1 \neq 0$, $e_1 \in$ V and let $f_1 = J(e_1)$. Let $L(e_1,f_1)$ denote the linear subspace of V spanned by e_1 and f_1. Then $L(e_1,f_1)$ is J invariant and since $J^4 = I$, there exists V_2 such that $JV_2 = V_2$ and

$$V = L(e_1,f_1) \oplus V_2 .$$

Identifying $L(e_1,f_1)$ with $\mathbb{C}$ as a real vector space by

$$ae_1 + bf_1 \longrightarrow a + bi$$

we can solve our problem by induction.

Henceforth (V,J) *will be called a complex vector space and* J *will be called the complex structure.*

Let us now consider the complex torus V/L, where (V,J) is our complex vector space. It is well known that the 1-forms $dx_1, dy_1, \ldots, dx_m, dy_m$ are a basis of $H^1(V/L,\mathbb{R})$, where $V = x_1e_1 + y_1f_1 + \ldots + x_me_m + y_mf_m$ and $Je_i = f_i$ and $Jf_i = -e_i$, $i = 1,\ldots,m$. If $V^* = H^1(V/L,\mathbb{R})$, then

$$H^*(V/L,\mathbb{R}) = \Lambda(V^*) .$$

Viewing V/L as a Lie group, we may identify the tangent space to V/L at the identity with V and V^* may be identified with the dual space to V. Hence, J induces J^* on V^* such that $(J^*)^2 = -I$. Thus a complex structure on a torus V/L is equivalent to an automorphism J^* of $\mathcal{F}_2(V^*)$ such that modulo the center of $\mathcal{F}_2(V^*)$, $(J^*)^2 = -I$.

Now let S be a compact Riemann surface of genus $m > 0$. (Topologically, S is a 2-sphere with m handles.) The classical facts about the cohomology ring $H^*(S,\mathbb{R})$ easily imply that N(S) is isomorphic to $N_{2m+1}(\mathbb{R})$. The orientation of S then determines an orientation of $N_{2m+1}(\mathbb{R})$. Let J(S) be the automorphism of $N_{2m+1}(\mathbb{R})$ induced by the complex structure on S, then J(S) acts trivially on $\mathfrak{z}$, the center of N(S), and if J_1 denotes the action of J(S) on $N(S)/\mathfrak{z}$, then $(J_1)^2 = -I$. Finally

$$[g, J(S)\, g\,] > 0 \qquad\qquad g \notin \mathfrak{z},\ g \in N(S)$$

where $[a,b] = aba^{-1}b^{-1}$.

Definition: Let $N_{2m+1}(\mathbb{R})$ be an oriented $\mathbb{R}$-Heisenberg group and let J be an automorphism of $N_{2m+1}(\mathbb{R})$ satisfying all the above conditions. We will call J a positive definite CR structure.

We are almost ready to define the concept of a Jacobi variety, but it will be convenient to make a slight detour in order to first define the concept of a dual torus.

Let V be an n-dimensional $\mathbb{R}$-vector space and let L be a discrete subgroup of V such that V/L is compact or a torus. Let V^* be the dual vector space to V and let $L^* \subset V^*$ be the subset of V^* such that $\ell^* \in L^*$ if and only if $\ell^*(L) \subset \mathbb{Z}$, where $\mathbb{Z}$ denotes the integers. One verifies that L^* is a discrete subgroup of V^* such that V^*/L^* is a torus which we will denote by $(V/L)^*$. We call L^* the dual lattice to L and $V^*/L^* = (V/L)^*$ the dual torus to V/L. Since $(L^*)^* = L$, we have $((V/L)^*)^* = V/L$.

Notice that $L \subset V$ determines a unique rational vector space $V(\mathbb{Q})$ such that

$$L \subset V(\mathbb{Q}) \subset V$$

and $L^* \subset V^*(\mathbb{Q}) \subset V^*$. Clearly, $V^*(\mathbb{Q})$ can be identified with $(V(\mathbb{Q}))^*$. We have already described $H^*(V/L,\mathbb{R})$ as $\wedge(V^*)$. Let $\mathcal{I}_2 = \sum_{i \leq 2} H^i(V/L, \mathbb{R})$ and consider $G(\mathcal{I}_2)$, the normal subgroup of $N(\wedge(V^*))$ determined by the ideal $\mathcal{I}_2$. Then one verifies that

$$N(\wedge(V^*))/G(\mathcal{I}_2)\Gamma(V/L)$$

is the dual torus to V/L.

Now form $N(S)/\Gamma(S)\mathfrak{z}$, where $\mathfrak{z}$ is the center of $N(S)$. Then $N(S)/\Gamma(S)\mathfrak{z}$ is a torus with a complex structure J^* determined by the positive definite CR structure on $N(S)$ and $N(S)/\Gamma(S)\mathfrak{z}$ determine a unique rational form for $N(S)/\mathfrak{z}$. Let V/L be the complex dual torus to $N(S)/\Gamma(S)\mathfrak{z}$. The complex torus $(V/L,J)$ is called the Jacobi variety of S.

The Jacobi variety of S is related to S by two important mappings. The first is at the cohomology level; the second is at the manifold level.

Consider $N(S)$. Since this is a graded nilpotent group, there exists an $A \in \mathrm{Alt}^{\times}(V^*)$ such that $N(S) = N(A)$, where $V^* = N(S)/\mathfrak{z}$ and $\mathfrak{z}$ is the center of $N(S)$. Let V/L, the dual torus of $N(S)/\Gamma(S)\mathfrak{z}$, be the Jacobi variety of S. Now $\wedge(V^*) = H^*(V/L,\mathbb{R})$ and so we have a natural homomorphism $N(V/L) \longrightarrow \mathcal{F}_2(V^*)$ with kernel $G(\mathcal{I})$, where $\mathcal{I}$ is the ideal $\sum_{i \geq 3} H^i(V/\Gamma,\mathbb{R})$. *Since*

$$\mathcal{F}_2(V^*)/[\mathcal{F}_2(V^*),\mathcal{F}_2(V^*)] = N(S)/\mathfrak{z} = V^* .$$

There is a unique surjection $P : \mathcal{F}_2(V^*) \longrightarrow N(S)$ (of course, P is the polarization determined by A) such that

$$\begin{array}{ccc} \mathcal{F}_2(V^*) & \xrightarrow{P} & N(S) \\ & \searrow \quad \swarrow & \\ & V^* & \end{array}$$

is a commutative diagram. Let $J(V/L)$ be the automorphism of $\mathcal{F}_2(V^*)$ induced by J. Then $J(V/L)$ is the same automorphism of $\mathcal{F}_2(V^*)$ as that induced by the complex structure on V/L. The mapping P is the first mapping at the cohomology level that we sought.

It is natural to ask if there exists a complex analytic mapping f : S $\longrightarrow$ V/L such that

$$f^*: N(\Lambda(V^*)) \longrightarrow N(S)$$

equals the composition $N(V/L) \longrightarrow \mathcal{F}_2(V^*) \xrightarrow{P} N(S)$. The answer is yes and the mapping of is called the Jacobi imbedding. Working this out in detail would take us too far afield from our main object so we will have to refer the reader to any of the many standard texts (for instance, [7]) for the proof of this result.

Now a complex torus that is a Jacobi variety has the remarkable property of having sufficiently many meromorphic functions to separate points.

Definition: A complex torus (V/L,J) is called an Abelian variety if it has sufficiently many meromorphic functions to separate points.

Remarks: *Not every complex torus is an Abelian variety. Not every Abelian variety is a Jacobi variety.*

We will now formulate necessary and sufficient conditions for a complex torus to be an Abelian variety.

Let (V/L,J) be a complex structure and let $\mathcal{F}_2(V^*) = N(\Lambda(V^*)/\sum_{i\geq 3} H^i(V/L,\mathbb{R}))$ and let J^* be the automorphism of $\mathcal{F}_2(V^*)$ induced by J. Recall that $L \subset V(\mathbb{Q}) \subset V$ and let $V^*(\mathbb{Q})$ be the dual rational vector space to $V(\mathbb{Q})$ with $V^*(\mathbb{Q}) \subset V$. Then $\mathcal{F}_2(V^*(\mathbb{Q})) \subset \mathcal{F}_2(V^*)$.

Definition: We call $P \in \mathrm{Pol}^{\times}(V^*)$ rational if the kernel P in $\mathcal{F}_2(V^*)$ is the closure of a subspace of $\mathcal{F}_2(V^*(\mathbb{Q}))$.

We can now state the fundamental theorem of Abelian varieties. Again we will have to leave a proof to outside sources such as [7]. Nilpotent proofs can be found in [8] and [3].

Theorem: A necessary and sufficient condition for (V/L,J) to be an Abelian variety is that there exists a rational polarization $P: \mathcal{F}_2(V^*) \longrightarrow N_{2m+1}(\mathbb{R})$, where $N_{2m+1}(\mathbb{R})$ is oriented, such that

1) The kernel of P is J^* invariant.

2) The automorphism that J^* induces on $N_{2m+1}(\mathbb{R})$ is a positive definite CR structure.

Definition: If $(V/L,J)$ is an Abelian variety, J is called a Riemann matrix. If P satisfies the above theorem, (J,P) will be called a Riemann pair.

It should be remarked that for fixed J there may be many rational polarizations $\{P\}$ such that (J,P) is a Riemann pair for $P \in \{P\}$. Also for each fixed P there may be many complex structures $\{J\}$ such that (J,P) is a Riemann pair for $J \in \{J\}$.

3. Morphisms of Abelian Varieties and the Structure of Riemann Matrices

Let $(V_1/L_1,J_1)$ and $(V_2/L_2,J_2)$ be Abelian varieties and let $f:V_1/L_1 \longrightarrow V_2/L_2$ be a complex analytic mapping. Then $f^*:N(V_2/\Gamma_2) \longrightarrow N(V_1/L_1)$ and

$$f^*J_2 = J_1 f^* .$$

We will call f or, by abuse of language, f^*, a morphism of the Abelian varieties.

Let End($\mathbb{A}$) be the ring of morphisms of an Abelian variety $\mathbb{A} = (V/L,J)$. Let

$$\text{End}(\mathbb{A}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{M}(J)$$

and call $\mathcal{M}(J)$ the rational multiplier algebra of J. Notice that $\mathcal{M}(J)$ depends on the rational structure of V and on J, but not on the lattice L. Clearly, $\mathcal{M}(J)$ is actually a *representation* of a rational associative algebra. Since $\mathcal{M}(J) \supset \mathbb{Q}$, $\mathcal{M}(J)$ is never trivial.

Let (J,P) be a Riemann pair and let A be the alternating form corresponding to P. Then A determines an isomorphism $A:V(\mathbb{Q}) \longrightarrow V^*(\mathbb{Q})$. For $M \in \mathcal{M}(J)$, one verifies [6] that $A^{-1}M^* A \in \mathcal{M}(J)$ and hence

$$\sigma:M \longrightarrow \sigma(M) = A^{-1}M^*A$$

is an involution of $\mathcal{M}(J)$.. This involution, called the Rosati involution , is also positive; i.e., the trace MM^* is positive if $M \neq 0$.

Let us now state a lemma due to Poincare′ that will enable us to completely structure $\mathcal{M}(J)$.

Poincare′ Lemma: Let (V/L,J) be an Abelian variety and let $V_1(\mathbb{Q}) \subset V(\mathbb{Q})$ be such that $V_1 = V_1(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \subset V$ is J invariant. Then there exists $V_2(\mathbb{Q}) \subset V(\mathbb{Q})$ such that

1) $V(\mathbb{Q}) = V_1(\mathbb{Q}) \otimes V_2(\mathbb{Q})$
2) $V_2 = V_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \subset V$ is J invariant.
3) If L_2 is a lattice in $V_2(\mathbb{Q})$, then $(V_2/L_2, J\,|\,V_2)$ is an Abelian variety.

Remark: The existence of $V_1(\mathbb{Q})$ is equivalent to $\mathcal{M}(J)$ containing a proper projection operator or idempotent.

We may find $V_2(\mathbb{Q})$ as follows. Let (J,P) be a Riemann pair and let G be the subgroup of $N_{2m+1}(\mathbb{R})$ generated by $P(V_1)$. Let $\mathscr{C}(G)$ denote the centralizer of G. Then $\mathscr{C}(G)/\mathscr{Z}$, where $\mathscr{Z}$ is the center of $N_{2m+1}(\mathbb{R})$, will be V_2. A proof of these assertions can be found in [3], Chapter III.

Remark: Clearly, the Poincare′ Lemma implies that $\mathscr{M}(J)$ is completely reducible and so is semi-simple.

We say that J is $\mathbb{Q}$-irreducible if $\mathscr{M}(J)$ has no non-trivial projections. Clearly, the Poincare′ Lemma implies that

$$V = V_1 \oplus \ldots \oplus V_n$$

where $J(V_i) = V_i$ and if $J_i = J \mid V_i$, then J_i is $\mathbb{Q}$-irreducible, all i. We say that J_i and J_j are $\mathbb{Q}$-equivalent if there exists $D \in \mathscr{M}(J)$ such that

$$D: V_i \longrightarrow V_j$$

and $DJ_i = J_jD$. We may group together all the $\mathbb{Q}$-equivalent J_i's and change the indexing to write

$$J = \sum_1^k m_i J_i \qquad m_i \in \mathbb{Z}^+ .$$

We call m_i the multiplicity of J_i. It follows that

$$\mathscr{M}(J) = \oplus \sum_{i=1}^k \mathscr{M}(m_i J_i) .$$

Now, it is easily seen that if J_i is irreducible, then $\mathscr{M}(J_i)$ is a representation ρ of a division algebra $\mathscr{D}_i$. Further, $\mathscr{M}(m_iJ_i)$ is the $m_i \times m_i$ matrix algebra over $\rho(\mathscr{D}_i)$. Thus to determine all Riemann matrices J, Albert had to first solve the following algebraic problems.

1) Determine the set Δ of all rational division algebra with a positive involution.

2) For $\mathscr{D} \in \Delta$ determine the set of all positive involutions.

Since the solution to these algebraic problems have many good expositions [1], we will just pull the algebraic results out of the hat as we need them. We adopt the following language that has become customary in this subject. Let $\mathscr{D} \in \Delta$ and, let k be the center of $\mathscr{D}$, and let σ be a positive involution of $\mathscr{D}$. Then $\sigma(k) = k$ and $\sigma \mid k$ is a positive involution. $\mathcal{D}$ is said to be of the *first kind* if $\sigma(k) = k$, all $k \in \mathcal{k}$ and if this fails, $\mathcal{D}$ is said to be of the *second kind*.

In these notes, we will not discuss the problem of $\mathbb{Q}$-irreducible Riemann matrices, but discuss the following simpler problem.

Main Problem: Let $\mathcal{D} \in \Delta$ and let ρ be a right $\mathbb{Q}$ representation of $\mathcal{D}$. Find all Riemann matrices J such that $\mathcal{M}(J) = \rho(\mathcal{D})$.

We will now outline our approach to this problem:

By the general representation theory, we know that every right representation ρ of $\mathcal{D}$ that could be a candidate for an irreducible J can be considered as pr, where r is the right regular representation of $\mathcal{D}$ over $\mathbb{Q}$ and $p \in \mathbb{Z}^+$. Assume ρ acts on the $\mathbb{Q}$-vector space V and form $\mathcal{F}_2(V)$, noting that ρ induces a representation of $\mathcal{D}$ as morphisms of $\mathcal{F}_2(V)$ that we will also denote by ρ. We next determine all $P \in \mathrm{Pol}^{\times}(V)$ such that if A is the alternating form corresponding to P then

$$A^{-1}\rho^{*}(d)A = \rho(\sigma(d)) \qquad d\in\mathcal{D},\sigma\in\{\sigma\}$$

where $\{\sigma\}$ is the set of positive involutions of $\mathcal{D}$. We let $\mathcal{A}(\mathcal{D},\rho,\sigma)$ denote the set of such polarizations. In other words, we first find the polarizations that can be candidates for a Rosati involution.

For each $P \in \mathcal{A}(\mathcal{D}, \rho,\sigma)$, we produce a Riemann matrix J(P) such that

1) $\mathcal{M}(J(P)) = \rho(\mathcal{D})$

2) (J(P),P) is a Riemann pair.

Finally, from J(P) we determine all Riemann matrices $\{J\}_P$ such that if $J \in \{J\}_P$:

1) $\mathcal{M}(J) \supset \rho(\mathcal{D})$

2) (J,P) is a Riemann pair.

4. Riemann Matrices Whose Multiplier Algebras are Totally Real Fields

The simplest examples of division algebras with positive involution are the totally real fields with the identity mapping as positive involution. Indeed, the identity mapping is the only positive involutions for totally real fields. Recall that k is totally real, $[k:\mathbb{Q}] = h$, if and only if k has h distinct isomorphisms into $\mathbb{R}$; or, if and only if the regular representation r of k over $\mathbb{Q}$ is diagonalizable over $\mathbb{R}$.

Assume for the rest of this section that k is totally real, $[k:\mathbb{Q}] = h$, and r is the regular representation of k over $\mathbb{Q}$. Up to rational equivalence, we may restrict ourselves to representations ρ of the form qr, $q \in \mathbb{Z}^+$; i.e., to multiples of the regular representation. Let $V(\mathbb{Q})$ be the $\mathbb{Q}$-vector space for the representation ρ. Then dim $V(\mathbb{Q}) = hq$. But $V(\mathbb{Q})$ can also be considered as a k-vector space by defining

$$kv = \rho(k)v \qquad k \in k,\ v \in V(\mathbb{Q}).$$

As a k-vector space we will denote $V(\mathbb{Q})$ by $V(k)$. Of course, the k-dimension of $V(k)$ is q.

We will now solve the problem of determining all polarizations of $\mathcal{F}_2(V(\mathbb{Q}))$ that induce the positive involution on k. For this argument, it will be convenient to adopt the following notation: For $A \in \mathrm{Alt}(V)$, let $\pi(A) \in \mathrm{Pol}(V)$ be the polarization of $\mathcal{F}_2(V)$ corresponding to A.

Let $A \in \mathrm{Alt}^\times(V(k))$ and let $\pi = \pi(A) \in \mathrm{Pol}^\times(V(k))$. Let $t : k \longrightarrow \mathbb{Q}$ be the trace mapping and set $B = t \circ A$. Then $B \in \mathrm{Alt}^\times(V(\mathbb{Q}))$ and $\pi' = \pi(B) \in \mathrm{Pol}^\times(V(\mathbb{Q}))$. Let x,y $\in V(k)$ and let $a \in k$, then

$$\rho^*(a)B(x,y) = B(x,ay) = t(A(x,ay)) = t(aA(x,y))$$

and

$$B\rho(a)(x,y) = B(ax,y) = t(A(ax,y)) = t(aA(x,y)).$$

Thus

$$B^{-1}\rho^*(a)B = \rho(a)\ .$$

Let $\mathcal{A}(k,\rho,\sigma)$, where σ is the identity mapping, denote all polarization $\pi(B)$ such that

$$B^{-1}\rho^*(a)B = \rho(\sigma(a)) = \rho(a)\ .$$

Then the image of $\mathrm{Pol}^\times(V(k))$ under t in $\mathrm{Pol}^\times(V(\mathbb{Q}))$ is contained in $\mathcal{A}(k,\rho,\sigma)$.

We will prove that $t(\mathrm{Pol}^{\times}(V(k)) = \mathscr{A}(k,\rho,\sigma)$.

Suppose $\pi' \in \mathrm{Pol}^{\times}(V(\mathbb{Q}))$, $\pi' = \pi'(B)$, $B \in \mathrm{Alt}^{\times}(V(\mathbb{Q}))$, and that $\pi' \in \mathscr{A}(k,\rho,\sigma)$ or

$$B^{-1}\rho^{*}(a)B = \rho(a) .$$

The equation is equivalent to $B(ax,y) = B(x,ay)$, $a \in k$, $x,y \in V(k)$. Then the mappings

$$\theta_{ij}: a \longrightarrow B(ar_i, v_j) \qquad a \in k$$

are $\mathbb{Q}$-linear mappings of k to $\mathbb{Q}$, where $v_1,\ldots,v_q$ define a basis of $V(k)$. Since the trace form is non-singular, there exists $\xi_{ij} = k$ such that

$$\theta_{ij}(a) = t(\xi_{ij}a) \qquad i \leq i,j \leq q .$$

Since B is alternating, $\xi_{ij} = -\xi_{ji}$. Let $x = \sum_1^q a_i v_i$ and $y = \sum_1^q b_j v_j$ where $a_i, b_j \in k$. Then

$$B(v,y) = t(\sum_{i,j} \xi_{ij}(a_i b_j - a_j b_i)) .$$

Let

$$A = \sum \xi_{ij}(a_i b_j - a_j b_i) .$$

Then, $A \in \mathrm{Alt}^{\times}(V(k))$ and $B = t \circ A$ and we have proven that $t(\mathrm{Pol}^{\times}(V(k)) = \mathscr{A}(k\rho,\sigma)$.

Thus the commutative diagram

$$\begin{array}{ccc} \mathscr{F}_2(V(k)) & \xrightarrow{\ \pi\ } & N_{2q+1}(k) \\ \uparrow p & & \downarrow t \\ \mathscr{F}_2(V(\mathbb{Q})) & \longrightarrow & N_{2qh+1}(\mathbb{Q}) \end{array}$$

enables us to identify $\mathscr{A}(k,\rho,\sigma)$ and $\mathrm{Pol}^{\times}(V(k))$.

Let us now consider the four groups that appear in the above diagram. $\mathscr{F}_2(V(\mathbb{Q}))$ and $N_{2qh+1}(\mathbb{Q})$ are rational algebraic groups, and therefore we may consider the group of real points of these algebraic groups, which we will denote by $\mathscr{F}_2(V(\mathbb{R}))$ and $N_{2qh+1}(\mathbb{R})$, respectively. In Section 1, we saw that $\mathscr{F}_2(V(k))$ and $N_{2q+1}(k)$ are the k points of rational algebraic groups; or, equivalently, are defined over $\mathbb{Q}$ Hence by reducing the field to $\mathbb{Q}$ we may form $\mathbb{Q}$-algebraic groups which are isomorphic to $\mathscr{F}_2(V(k))$ and $N_{2q+1}(k)$ and form the group of real points of these $\mathbb{Q}$-algebraic groups, which we will denote by $\mathscr{F}_2(V(k))_{\mathbb{R}}$ and $N_{2q+1}(k)_{\mathbb{R}}$, respectively. Let $\chi_i : k \longrightarrow \mathbb{R}$ $i=1,\ldots,h$ be distinct isomorphisms of k into $\mathbb{R}$

We claim that

$$N_{2q+1}(\mathbb{k})_{\mathbb{R}} = \prod_{1}^{h} N_{2q+1}(\mathbb{R},\chi_i)$$

where $N_{2q+1}(\mathbb{R},\chi_i)$ is isomorphic to $N_{2q+1}(\mathbb{R})$, i=1,....,k.

The above assertion may be seen as follows:

$$N_{2q+1}(\mathbb{k}) = \begin{pmatrix} 1 & k_1 \ldots k_q & k \\ & \ddots \quad 0 & k_{q+1} \\ & \ddots & \vdots \\ & & k_{2q} \\ 0 & & 1 \end{pmatrix} \quad k_\alpha \epsilon \mathbb{k},\ k \epsilon \mathbb{k},\ \alpha = 1,\ldots,2q.$$

If we reduce the field to $\mathbb{Q}$ we obtain

$$r(\mathbb{Q})(N_{2q+1}(\mathbb{k})) = \begin{pmatrix} r(1) & r(k_1)\ldots r(k_q) & r(k) \\ & \ddots \quad r(0) & r(k_{q+1}) \\ & \ddots & \vdots \\ r(0) & & r(k_{2q}) \\ & & r(1) \end{pmatrix}$$

where the right side above is a matrix of h+h matrices. Now, as in Section 1, let A ϵ GL(h,$\mathbb{R}$) be such that

$$A^{-1} r(k) A = D(k) \qquad k \epsilon \mathbb{k}$$

where

$$D(k) = \begin{pmatrix} \chi_1(k) & & 0 \\ & \ddots & \\ 0 & & \chi_h(k) \end{pmatrix}$$

Let

$$(q+2)A = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}$$

where the matrix on the right side is a (q+2) × (q×2) matrix of h×h matrices. Then

$$(q+2)A^{-1}\, r(\mathbb{Q})(N_{2q+1}(k))(q+2)A = \begin{pmatrix} D(1) & D(k_1)\ldots D(k_q) & D(k) \\ & D(0) & \vdots \\ & & D(k_{2q}) \\ D(0) & & D(1) \end{pmatrix}$$

It is now easily seen that we may rearrange rows and columns of the above matrix to obtain

$$\oplus\sum_{i=1}^{h} \begin{pmatrix} \chi_i(1) & \chi_i(k_1)\ldots\chi_i(k_q) & \chi_i(k) \\ \ddots & & \chi_i(k_{q+1}) \\ & \chi_i(0) & \vdots \\ & & \chi_i(k_{2q}) \\ \chi_i(0) & & \chi_i(1) \end{pmatrix}$$

This proves our assertion and explains that the notation $N_{2q+1}(\mathbb{R},\chi_i)$ stands for the 2q+1 $\mathbb{R}$-Heisenberg with k imbedded in $\mathbb{R}$ by the isomorphism χ_i,

Now $t: N_{2q+1}(k) \longrightarrow N_{2qh+1}(\mathbb{Q})$ uniquely extends to a homomorphism $T: \prod N_{2q+1}(\mathbb{R},\chi_i) \longrightarrow N_{2qh+1}(\mathbb{R})$. Further, T restricted to the center of $N_{2q+1}(\mathbb{R},\chi_i)$ is an isomorphism for each i. If $N_{2qh+1}(\mathbb{R})$ is oriented, we may use T to orient each $N_{2q+1}(\mathbb{R},\chi_i)$ by requiring that the induced isomorphism on each center is orientation preserving. We will then say that $\prod N_{2q+1}(\mathbb{R},\chi_i)$ is coherently oriented.

We are now going to find all complex structures J on $V(\mathbb{R})$ such that for $P \in \mathscr{A}(k,\rho,\sigma)$, (J,P) is a Riemann pair and $\rho(k)\, J = J\rho(k)$, $k \in k$.

Assume J satisfies the two conditions above. Since J is an automorphism of $\mathscr{F}_2(V\mathbb{R}))$ that commutes with $\rho(k)$, it follows that J determines an automorphism of $\mathscr{F}_2(V(k))_{\mathbb{R}}$. By hypothesis, J preserves the real closure of the kernel of $P = t \circ \pi \circ p$. Hence, J preserves the kernel of $\mathscr{F}_2(V(k))_{\mathbb{R}} \longrightarrow N_{2q+1}(k)_{\mathbb{R}}$ and so J induces an automorphism J_1 of $N_{2q+1}(k)_{\mathbb{R}}$. Because k operates on each $N_{2q+1}(\mathbb{R},\chi_i)$ modulo its center as $\chi_i(k)$ (i.e., as a diagonal matrix) and since χ_i, i=1,...,h, are all distinct, it follows that J_1 must leave each $N_{2q+1}(\mathbb{R},\chi_i)$ invariant. If J is a positive definite CR structure relative to $N_{2qh+1}(\mathbb{R})$, it follows that $J_1 \mid N_{2q+1}(\mathbb{R},\chi_i)$, all i, is a positive definite CR structure. Hence, we have that if J satisfies our two conditions, then J may be written as $\prod^{h} J_i$ where each J_i is a positive definite CR structure for $N_{2q+1}(\mathbb{R},\chi_i)$, i=1,...,h.

Now let J_i be a complex structure automorphism of $N_{2q+1}(\mathbb{R},\chi_i)$ that determines a positive definite CR structure on $N_{2q+1}(\mathbb{R},\chi_i)$ $i=1,\ldots,h$. Let $J_1 = \prod^{h} J_i$. Then J_1 is an automorphism of $\prod^{h} N_{2q+1}(\mathbb{R},\chi_i)$ that acts trivially on the center of this group. Hence, $T \circ J_1$ is an automorphism of $N_{2hq+1}(\mathbb{R})$ that is a positive definite complex structure. Clearly, there exists an automorphism J of $\mathcal{F}_2(V(\mathbb{R}))$ that commutes with $\rho(\mathcal{k})$ and lifts $T \circ J_1$.

Thus, if $\mathcal{J}$ is the set of positive definite CR structures in $N_{2q+1}(\mathbb{R})$, we have that the set of all complex structures J on $V(\mathbb{R})$ such that for $P \in \mathcal{A}(\mathcal{k},\rho,\sigma)$, (J,P) is a Riemann pair and $\rho(k)J = J\rho(k)$, $k \in \mathcal{k}$ is of the form $\prod^{h} \mathcal{J}$.

5. The Involution Problem for Division Algebras Of the First Kind (Part I)

Let $\mathscr{D}$ be a division algebra over $\mathbb{Q}$ that has a positive involution of the first kind. Then by the algebraic theory $\mathscr{D}$ may be described as follows: Let k be the center of $\mathscr{D}$. Then k is a totally real number field and we may assume $[k,\mathbb{Q}] = h$. If $\mathscr{D}$ does not equal k, then $\mathscr{D}$ is a quaternion division algebra $\mathbb{K} = \mathbb{K}(a,b)$ over k defined as a 4-dimensional k-vector space with k basis 1, i, j, k satisfying

$$i^2 = a \quad j^2 = b \quad k^2 = -ab$$

$$ij = -ji = k; \; jk = -kj = -bi; \; ki = -ik = -aj$$

where $a,b \in k$.

Let L be any subfield of $\mathbb{K}$ and consider $\mathbb{K}$ as a left L-vector space. The right regular representation of $\mathbb{K}$ over L is given by

$$r_L : \mathbb{K} \longrightarrow \mathrm{Hom}_L(\mathbb{K},\mathbb{K})$$

as L-vector spaces, where $r_L(\delta_1)\delta_2 = \delta_2\delta_1$, $\delta_1,\delta_2 \in \mathbb{K}$.Let

$$N_L(\delta) = \det r_L(\delta) .$$

In particular, we define the norm of δ, $N(\delta)$, $\delta \in \mathbb{K}$, by $N(\delta) = N_L(\delta)$ where L is the maximal commutative subfield $k(i)$, $i^2 = a \in k$. Explicitly, if $\delta = x + yi + zj + tk \in \mathbb{K}$, then $N(\delta) = x^2 - ay^2 - bz^2 + abt^2$. The algebraic theory tells us that because $\mathbb{K}$ is a division algebra $N(\delta) \neq 0$, unless $\delta = 0$. If we identify k with $k \cdot 1 \subset \mathbb{K}$, we have that $N : \mathbb{K}^\times \longrightarrow k^\times$ is a homomorphism, where $\mathbb{K}^\times$ is the multiplicative subgroup of $\mathbb{K}$. We next note that $N(\delta) = \delta\sigma(\delta)$, $\delta \neq 0$, implies that $\sigma(\delta) = \delta^{-1} N(\delta)$. Since $N(\delta)$ is central in $\mathbb{K}$, we have

$$\sigma(\delta_1\delta_2) = \delta_2^{-1}\delta_1^{-1} N(\delta_1\delta_2) = \sigma(\delta_2)\sigma(\delta_1) .$$

Explicitly, if $\delta = x + iy + jz + kt$, then

$$\sigma(\delta) = x - iy - jz - kt .$$

We will use r to denote the right regular representation of $\mathbb{K}$ over k. Similarly, we