Dario Graffi (Ed.)

Non-Linear Mechanics

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FONDAZIONE CIME ROBERTO CONTI

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Non-Linear Mechanics

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NON-LINEAR MECHANICS

(Coordinatore: D. GRAFFI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

L. CESARI

NONLINEAR ANALYSIS

Corso tenuto a Bressanone dal 4 al 13 giugno 1972

NONLINEAR ANALYSIS

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We present here outlines of a series of research papers in nonlinear analysis, all centered on the concept of bifurcation equation, and mostly relying on methods and ideas of functional analysis. While we do not claim completeness, we attempt to give a fair view on relevant trends in the field. In each of the following short presentations we list a few bibliographical references.

Lamberto Cesari

CONTENTS

Lamberto Cesari

I. PERTURBATION PROBLEMS FOR PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

The approach described below for the problems under consideration was developed in the years 1952-60 and was later framed in the general method we will describe in Part III. Nevertheless the results obtained by this approach and the detailed technique which was correspondingly developed required a presentation of their own.

Let us consider a system of ordinary differential equations of the form

$$du/dt = A(\varepsilon)u + \varepsilon f(t,u,\varepsilon) , \tag{1}$$

where ε is a small parameter, A is an nxn real or complex constant matrix whose elements may depend on ε, $u = \mathrm{col}(u_1,\dots,u_n)$, $f = \mathrm{col}(f_1,\dots,f_n)$, and $f(t,u,\varepsilon)$ is periodic in t of some period $L = 2\pi/\omega$, or is independent of t, and then (1) is autonomuos and L is arbitrary. Instead of (1) we may consider the analogous system

$$du/dt = A(\varepsilon)u + \varepsilon f(t,u,\varepsilon) + F(t) , \tag{1'}$$

where $F(t)$ is a given periodic function of t. We are interested in the possible periodic solutions of system (1), or (1'). They are sometimes called periodic oscillations, and indeed, they represent actual oscillatory phenomena of corresponding physical systems. The periodic solutions of system (1), when this is autonomous, are often called cycles.

System (1) is often thought of as a perturbation of the linear system with constant coefficients $du/dt = A(0)u$. Its periodic solutions, if any, may have large amplitude, even if ε is small. The term $F(t)$ in (1') is often denoted as a "large" forcing term.

Systems (1) and (1') (for $\varepsilon \neq 0$) are not any easier to handle because ε is "small." The phenomena which (1) and (1') may represent (resonance, nonresonance, harmonic oscillations, subharmonic, higher harmonics, cycles' frequency depending on ε for autonomous systems, entrainement of frequency, or locking of frequency, stability of the corresponding solutions) are varied and complex (cfr., e.g., L. Cesari [10] for some references).

We shall now mention briefly a simple process for the determination and study of periodic solutions of systems (1), or (1'), and some of the related phenomena. In Part III we shall frame this process in a general method for boundary value problems.

To begin with, let us first consider the case of a system (1) with $A = 0$, or

$$du/dt = \varepsilon f(t,u,\varepsilon) , \qquad u = (u_1,\dots,u_n) , \tag{2}$$

where $f(t,u,\varepsilon) = (f_1,\ldots,f_n)$ is, say, defined, continuous and periodic in t of period $L = 2\pi/\omega$ in the domain $-\infty < t < +\infty$, $|u| \leq R$, $|\varepsilon| \leq \varepsilon_0$, for some R, $\varepsilon_0 > 0$.

Let S denote the space of all continuous vector functions $\varphi(t) = (\varphi_1,\ldots,\varphi_n)$ periodic of period L, and let us take in S the uniform topology. For $\varphi \epsilon S$ let $P\varphi = (P\varphi_1,\ldots,P\varphi_n) = L^{-1}\int_0^T \varphi(t)dt$ denote the mean value of φ. Thus, $P\varphi$ is the constant vector whose components $P\varphi_j$ are the mean values of the components φ_j of φ. Also, P can be thought of as an operator P: $S \to S$, and as such P is linear, idempotent (that is, $PP = P$), and is a projector operator.

Let $c = (c_1,\ldots,c_n)$ denote any arbitrary constant vector with $|c| \leq r < R$, $j = 1,\ldots,n$, for some fixed r, $0 < r < R$. Let S_c denote the set of all $\varphi \epsilon S$, $\varphi = (\varphi_1,\ldots,\varphi_n)$ with $P\varphi = c$ and $|\varphi(t)| \leq R$. We consider now the transformation $\psi = T\varphi$ defined for every $\varphi \epsilon S_c$ by

$$\psi(t) = c + \varepsilon \int [f(\tau,\varphi(\tau),\varepsilon) - m]d\tau ,$$

$$m = Pf(\tau,\varphi(\tau),\varepsilon) . \tag{3}$$

Here m is the mean value of the periodic function $f(\tau,\varphi(\tau),\varepsilon)$; hence, $f(\tau,\varphi(\tau),\varepsilon) - m$ has mean value zero, and $\int$ in (3) denotes the unique primitive periodic of period L and mean value zero, so that $\psi \in S$, $P\psi = c$.

Since $|c| < r < R$, it is easily seen that, for $|\varepsilon|$ sufficiently small, we also have $|\psi(t)| \leq R$, and thus $T\colon S_c \to S_c$. It is easily seen also that S_c is a convex subset of S, and that $T(S_c)$ is compact. By Schauder fixed point theorem there is, therefore, at least one element $u \in S_c$ which is transformed by T into itself, that is, a fixed element $u = Tu$ of T, asy $u(t;c,\varepsilon)$. Relation (3) for $\varphi = \psi = u$, shows that u is a solution of the modified differential system

$$du/dt = \varepsilon[f(t,u,\varepsilon) - m], \qquad m = Pf(t,u(t),\varepsilon).$$

Moreover, if we assume that $f(t,u,\varepsilon)$ is Lipschitzian with respect to u in the domain $-\infty < t < +\infty$, $|u| \leq R$, $|\varepsilon| \leq \varepsilon_o$, then, again for $|\varepsilon|$ sufficiently small, T is a contractio, and thus the fixed element $u = Tu$ is unique and depends continuously on c and ε.

The question remains as to whether, again for $|\varepsilon|$ sufficiently small, we can determine $c = (c_1,\dots,c_n) = c(\varepsilon)$, $|c| \leq r$, so that $m = (m_1,\dots,m_n) = 0$. The equation $m = 0$, or $m_j = 0$, $j = 1,\dots,n$, is a first instance of what we shall denote the bifurcation equation, or determining equation. If $c(\varepsilon)$ is any solution of the bifurcation equation $m = 0$, then $u(t,c(\varepsilon),\varepsilon)$ is a periodic solution of the original differential system (2).

For this process and various modifications, one of which is mentioned below, we refer to L. Cesari [6]. A linear version of it was proposed by Cesari in [5], and variants and applications have been discussed by R. Gambill and J. K. Hale [17]. As mentioned, we shall see in Part III a more general process for boundary value problems.

Let us consider a periodic system (1) of period $T = 2\pi/\omega$ of which we want to determine solutions of some period in rational ratio with T. Let $\rho_1,\dots,\rho_n$ denote the n eigenvalues of A (which in general depend on ε), and let us separate those $\rho_1,\dots,\rho_\nu$ which, as $\varepsilon\to 0$, approach purely imaginary numbers $\rho_j(0) = i\tau_j = ia_j\omega/b_j$, $j = 1,\dots,\nu$, a_j, b_j integers, $b_j > 0$, $a_j \gtreqless 0$, from those $\rho_{\nu+1},\dots,\rho_n$ whose limits as $\varepsilon\to 0$ are complex or real numbers different from any number $ia\omega/b_0$, $b_0 = b_1\cdots b_\nu$, a integer, $a \gtreqless 0$. Here ν denote any integer $0 \leq \nu \leq n$, and the case $\nu = 0$ is particularly simple, and we shall not discuss it here. Correspondingly, we can reduce A in (1) to the form $\mathrm{diag}(A_1,A_2)$, A_1 with eigenvalues $\rho_1,\dots,\rho_\nu$, A_2 with eigenvalues $\rho_{\nu+1},\dots,\rho_n$. To simplify the present exposition we assume that all roots $\rho_1,\dots,\rho_n$ are simple, so that we can reduce A in (1) to the diagonal form $A = \mathrm{diag}(\rho_1,\dots,\rho_n)$. We shall denote by B the analogous matrix $B = \mathrm{diag}(i\tau_1,\dots,i\tau_\nu, \rho_{\nu+1},\dots,\rho_n)$.

Let S denote the space of all continuous vector functions $\varphi(t) = (\varphi_1,\dots,\varphi_n)$ periodic of period $2b_o\pi/\omega$, and let us take in S the uniform topology. Let P be defined now by $P\varphi = (P\varphi_1,\dots,P\varphi_\nu, 0,\dots,0)$. For the sake of simplicity let us assume here that $f(t,u,\varepsilon)$ is defined, continuous, and periodic in t of period $L = 2\pi/\omega$, in the domain $-\infty < t < +\infty$, $|u| \leq R$, $|\varepsilon| \leq \varepsilon_o$ for some R, $\varepsilon_o > 0$. Let $c = (c_1,\dots,c_\nu,0,\dots,0)$ denote any constant vector (with $c_1,\dots,c_\nu \neq 0$) and $|c| \leq r < R$. Let $z(t) = (c_1 e^{i\tau 1 t},\dots,c_\nu e^{i\tau\nu t},0,\dots,0)$. Let S_c denote the set of all $\varphi \epsilon S$, $\varphi = (\varphi_1,\dots,\varphi_n)$ with $P(e^{-Bt}\varphi) = c$ and $|\varphi(t)| \leq R$, so that $z \epsilon S_c$. We consider now the transformation $\psi = T\varphi$ defined for every $\varphi \epsilon S_c$ by

$$\begin{aligned} \psi(t) &= z(t) + \varepsilon\, e^{Bt} \int e^{-B\tau}[f(\tau,\varphi(\tau),\varepsilon) - D\varphi(\tau)]d\tau , \\ D &= \operatorname{diag}(d_1,\dots,d_\nu,0,\dots,0) , \quad Dc = P[e^{-B\tau} f(\tau,\varphi(\tau),\varepsilon)] . \end{aligned} \tag{4}$$

It has been shown in [6] that it is possible to choose the primitives in (4) so that ψ is periodic, $\psi \epsilon S$, $P[e^{-Bt}\psi] = P[e^{-Bt}z] = c$.

Since $|c| \leq r < R$, it can be proved (cfr. [6]) that, for $|\varepsilon|$ sufficiently small, we also have $|\psi(t)| \leq R$, and thus T: $S_c \to S_c$. It can also be proved (cfr. [6]) that S_c is a convex subset of S, and that $T(S_c)$ is compact. By Schauder fixed point theorem there is, therefore, at least one fixed element $u \epsilon S_c$, $u = Tu$, say

$u(t;c,\varepsilon)$. Relation (4) for $\varphi = \psi = u$ shows that u is a solution of the modified differential system

$$du/dt = (B - \varepsilon D)u + \varepsilon f(t,u,\varepsilon) ,$$

$$Dc = P[e^{-Bt} f(t,u(t),\varepsilon)] ,$$

$$D = \operatorname{diag}(d_1,\ldots,d_\nu, 0,\ldots,0) .$$

Again as before, if we assume that $f(t,u,\varepsilon)$ is Lipschitzian with respect to u in the domain $-\infty < t < +\infty$, $|u| \le R$, $|\varepsilon| \le \varepsilon_o$, then, for $|\varepsilon|$ sufficiently small, the map T: $S_c \to S_c$ is a contraction (cfr. [6]), and thus the fixed point $u = Tu$ in S_c is unique, and it can be proved to depend continuously upon c and ε.

The question remains as to whether, again for $|\varepsilon|$ sufficiently small, we can determine $c = (c_1,\ldots,c_\nu,0,\ldots,0) = c\ (\varepsilon)$, $|c| \le r$, so that $B - \varepsilon D = A$, that is,

$$ia_j\omega/b_j - \varepsilon\, d_j\, c_j = \rho_j(\varepsilon) , \qquad j = 1,\ldots,\nu .$$

The equation $B - \varepsilon D = A$ is the bifurcation or determining equation. If $c(\varepsilon)$ is any solution of the bifuration equation, $|c(\varepsilon)| \le r$, then $u(t,c(\varepsilon),\varepsilon)$ is a periodic solution of the original differential system (1).

The argument above remains the same for autonomous systems, with the difference that now $\omega = \omega(\varepsilon)$ is among the unknowns in solving the bifurcation equation, and, on the other hand, the phase in the solutions must remain undetermined, so that ν is still the number of essential unknowns.

Under sole conditions of continuity the existence of solutions to the bifurcation equation, and hence to the original problem, can be assured by the use of the concept of topological degree and of fixed point theorems. Actually, the following statement by C. Miranda has been shown to be relevant [6]. It is an equivalent form of Brouwer's fixed point theorem. This statement concerns vector valued continuous functions $F(z) = (F_1, \ldots, F_n)$ defined on an interval $C = [z = (z_1, \ldots, z_n) \mid |z^i| \leq R_i,\ i = 1, \ldots, n]$. If F_i has constant opposite signs on each of the sides $z^i = \pm R_i$ of C, then there is at least one point $z \epsilon C$ where $F(z) = 0$ (C. Miranda, Boll. Un. Mat. Ital. 3, 1941, 5-7).

For the case $\nu = 1$, which is the most usual one, much less is needed, since then the bifurcation equation reduces to a single quation in one real variable, and all is reduced to the verification that a function $F(z)$ of the real variable z has opposite signs at the end points of the interval $[-R, R]$.

For any $\nu \geq 1$ and f smooth, much more elementary considerations based on the implicit function theorem of calculus lead to simple and practical criteria for the existence and determination of $c = c(\varepsilon)$ for $|\varepsilon|$ small.

The following theorem (cfr. [6]), corresponding to the case $\nu = 1$, is a particularly simple statement for autonomous systems which can be derived from the above process.

Let us consider the system of first and second order differential equations

$$
\begin{aligned}
y_1'' + \sigma_1^2 y_1 &= \varepsilon\, f_1(y,y',\varepsilon)\ , \\
y_j'' + 2\alpha_j y_j + \sigma_j^2 y_j &= \varepsilon\, f_j(y,y',\varepsilon)\ , \qquad j = 2,\dots,\mu\ , \\
y_j'' + \beta_j y_j &= \varepsilon\, f_j(y,y',\varepsilon)\ , \qquad j = \mu+1,\dots,n\ ,
\end{aligned}
\tag{5}
$$

where $y = (y_1,\dots,y_n)$, $y' = (y_1',\dots,y_\mu')$, $f = (f_1,\dots,f_n)$, $1 = \nu \leq \mu \leq n$, where $\sigma_j(\varepsilon)$, $\alpha_j(\varepsilon)$, $\beta_j(\varepsilon)$ are real continuous functions of ε (or constants), $0 \leq \varepsilon \leq \varepsilon_o$, and $\sigma_j(0) > 0$, $j = 1,\dots,\mu$, $\beta_j(0) \neq 0$, $j = \mu+1,\dots,n$, and either $\alpha_j(0) \neq 0$, or $\alpha_j(0) = 0$, $\sigma_j(0) \not\equiv 0 \pmod{\sigma_1(0)}$, $j = 2,\dots,\mu$. We assume $a = b = 1$, $\sigma_1(0) = \omega_o = \tau_1$ and c is now a scalar. We assume that $f(y,y',\varepsilon)$ is Lipschitzian in y,y' and continuous with respect to ε for

$|y| \leq R$, $|y'| \leq R$, $0 \leq \varepsilon \leq \varepsilon_o$. For $\omega_{o1} \leq \omega \leq \omega_{o2}$, $0 < r_1 \leq \lambda' \leq \lambda \leq \lambda'' \leq r_2 < R$, $L = 2\pi/\omega$, let

$$P(\lambda,\omega) = (L\lambda)^{-1} \int_0^L f_1(\lambda\omega^{-1}\sin\omega t,0,\ldots,0,\lambda\cos\omega t,0,\ldots,0;0)\cos\omega t \, dt.$$

(I.i) If for some $\omega_{o1} < \omega_o < \omega_{o2}$, $\lambda' < \lambda_o < \lambda''$, we have $P(\lambda_o,\omega_o) = o$, and $P(\lambda',\omega_o)$, $P(\lambda'',\omega_o)$ have opposite signs, then there is an $\varepsilon_1 > 0$ such that for every ε, $0 \leq \varepsilon \leq \varepsilon_1$ system (5) has at least one periodic solution of the form

$$y_1(t,\varepsilon) = \lambda(\varepsilon)\omega^{-1}(\varepsilon) \sin\omega(\varepsilon)(t+\theta) + O(\varepsilon) ,$$

$$y_j(t,\varepsilon) = O(\varepsilon) , \qquad j = 2,\ldots,n,$$

for convenient $\omega(\varepsilon) \in [\omega_{o1},\omega_{o2}]$, $\lambda(\varepsilon) \in [\lambda',\lambda'']$, $\omega(0) = \omega_o$, $\lambda(0) = \lambda_o$, and the phase θ is of course arbitrary.

For details and proofs of this and other general existence theorems and criteria for periodic solutions for systems (1), periodic, or autonomous, we refer to Cesari [6]. See also [6] for extensions to types of linear systems more general than (1), extensions to the case where f satisfies weaker forms of the Lipschitz condition, and extensions to quasi periodic solutions.

For systems (1') containing a large forcing term $F(t) = (F_1,\ldots,F_n)$, say F periodic of period $L = 2\pi/\omega$ (F large in the sense that it does not vanish with ε), the existence of perturbation-type solutions requires in general that $P[e^{-Bt}F(t)] = 0$. In this situation, the linear system $u' = A(0)u$ has a periodic solution $U(t)$, and the change of variables $u = U(t) + v$ transforms the nonlinear system (1') into an analogous system (1) without forcing term.

In the discussion above the case $\nu = 0$ is rather elementary, and the determination of periodic solutions for the nonlinear case does not require the analysis of bifurcation equations.

If we consider $z(t)$ as the 0^{th} approximation $z^o(t)$ of the periodic solution of system (1), then the iterated scheme $z^{n+1} = Tz^n$, $n = 0,1,\ldots$, can be used in combination with the equation $B - \varepsilon D = A$ to get successive approximations of the functions $c = c(\varepsilon)$ and of the periodic solution $u(t,c(\varepsilon),\varepsilon)$ for $|\varepsilon|$ sufficiently small. Preliminary work has been done by J. K. Hale and R. Gambill [14-17,19-24] who studied in detail this method of successive approximations and discussed many applications of this method (cfr. particularly [17]). A relevant improvement in the method of successive approximations has been proposed recently by C. Banfi and G. Casadei [2-4].

The case where the matrix A in system (1) has multiple eigenvalues has been studied in detail by C. Imaz [30] by the approach described above.

As mentioned, J. K. Hale and R. A. Gambill discussed in [17] great many examples by the method above: e.g., the autonomous van der Pol equation

$$u'' + u = \varepsilon(1-u^2)u' ;$$

the nonlinear Mathieu equation with large forcing term

$$u'' + \sigma^2 u = A \cos 2\omega t + (Bx\cos 2\omega t + Cu^3) ;$$

the van der Pol equation with mild forcing term:

$$u'' + \sigma^2 u = \varepsilon(1-u^2)u' + \varepsilon\, p\omega\cos(\omega t + \alpha) ;$$

the generalized van der Pol equation

$$u'' + \sigma^2 u = \varepsilon(1-u^{2m})u' + \varepsilon\, p\omega\cos(\omega t + \alpha) ,$$

with m integer and large (almost square characteristic function); the system of two nonlinear Mathieu equations

$$u'' + \sigma_1^2 u = \varepsilon(Au + Bu \cos t + Cu^3 + Duv^2) ,$$

$$v'' + \sigma_2^2 v = \varepsilon(Ev + Fy \cos t + Gv^3 + Hu^2 v) ;$$

the autonomous system of two nonlinear equations

$$u'' + u - \varepsilon(1-u^2-v^2)u' = \varepsilon f_1(u,v,v') + \varepsilon g_1(u,u',v,v')v ,$$

$$v'' + 2v - \varepsilon(1-u^2-v^2)v' = \varepsilon f_2(u,u',v) + \varepsilon g_2(u,u',v,v')u ,$$

with

$$f_1(-u,v,v') = -f_1(u,v,v'), \; f_2(u,u',-v) = -f_2(u,u',v).$$

For instance, the latter system has, for $|\varepsilon|$ sufficiently small, two cycles given by

$$u = \lambda \sin(\omega t + \Phi) + O(\varepsilon) , \qquad v = O(\varepsilon) ,$$

$$\lambda = 2 + O(\varepsilon) , \qquad \omega = 1 + O(\varepsilon) ,$$

and

$$u = O(\varepsilon) , \qquad v = \lambda \sin(\omega t + \Phi) + O(\varepsilon) ,$$

$$\lambda = 2^{1/2} + O(\varepsilon) , \qquad \omega = 2^{1/2} + O(\varepsilon) .$$

J. K. Hale [21,24] proved, by the same method above, the existence of families of periodic solutions and of cycles for systems (1) under suitable symmetry relations.

Of a number of Hale's theorems concerning families of periodic solutions of systems (1), we report here only one concerning autonomous systems (under sole Lipschitz hypotheses as proved in [6]). This theorem guarantees the existence of a $(n - \mu + 2)$-parameter family of periodic solutions (cycles).

Let us consider the autonomous system

$$y_j'' + \sigma_j^2 y_j = \varepsilon f_j(y,y',\varepsilon) , \qquad j = 1,\ldots,\mu ,$$

$$y_j' = \varepsilon f_j(y,y',\varepsilon) , \qquad j = \mu + 1,\ldots,n , \qquad (6)$$

where $y = (y_1,\ldots,y_n)$, $y' = (y_1',\ldots,y_\mu')$, $f = (f_1,\ldots,f_n)$, where f is Lipschitzian with respect to y, y' and continuous in ε for $|y| \leq R$, $|y'| \leq R$, $0 \leq \varepsilon \leq \varepsilon_0$.

(I.ii) Let us assume, for f_1 only, that

$$f_1(0,y_2,\ldots,y_n,0,y_2',\ldots,y_\mu',\varepsilon) = 0 ,$$

and that either all f_j, $j = 1,\ldots,n$, are odd in $(y_1,\ldots,y_\mu)$, or f_1 is even and $f_2,\ldots,f_n$ are odd in $(y_2,\ldots,y_\mu,y_\mu')$. Suppose $\sigma_1(\varepsilon) > 0$, $j = 1,\ldots,\mu$, are continuous functions of ε (or constants), with $\sigma_j(0) \not\equiv 0(\mathrm{mod}\ \sigma_1(0))$, $j = 2,\ldots,\mu$. Take $\omega_0 = \sigma_1(0)$, and let r_2 be any number $0 < r_2 < R$. Then there exists an ε_1, $0 < \varepsilon_1 < \varepsilon_0$, such that, for all ε, λ_1, $\eta_1,\ldots,\eta_{n-\mu}$, $0 < \lambda_1$, $\eta_1,\ldots,\eta_{n-\mu} \leq r_2$, $0 \leq \varepsilon \leq \varepsilon_1$, system (6) has a real periodic solution of the form

$$y_1(t,\varepsilon) = \lambda_1\omega^{-1}\sin\omega t + O(\varepsilon)\ , \text{ or } y_1(t,\varepsilon) = \lambda_1\omega^{-1}\cos\omega t + O(\varepsilon)\ ,$$

$$y_j(t,\varepsilon) = O(\varepsilon)\ , \qquad j = 2,\ldots,\mu\ ,$$

$$y_j(t,\varepsilon) = \eta_{j-\mu} + O(\varepsilon)\ , \qquad j = \mu+1,\ldots,n\ ,$$

where y_1 is odd or even in t, $y_2,\ldots,y_\mu$ are odd, $y_{\mu+1},\ldots,y_n$ are even, where $\omega = \omega(\varepsilon,\lambda_1,\eta_1,\ldots,\eta_{n-\mu})$ is a continuous function of the same parameters, $\omega = \omega_o$ for $\varepsilon = 0$, and t can be replaced by $t + \theta$, the phase θ being arbitrary.

The following examples, all derived from the statement above or by analogous theorems (cfr. [21,24,6]), may illustrate the situation. For instance the simple equation

$$u'' + u = \varepsilon f(u,u')\ ,$$

with $f(0,0) = 0$, and either $f(u,-u) = f(u,u')$, or $f(-u,u') = -f(u,u')$, has a family of cycles of the form $u = \lambda\omega^{-1}\cos(\omega t + \phi) + O(\varepsilon)$, or $u = \lambda\omega^{-1}\sin(\omega t + \phi) + O(\varepsilon)$, with $\omega = \omega(\lambda,\varepsilon) = 1 + O(\varepsilon)$, λ, ϕ arbitrary, $|\varepsilon|$ sufficiently small. We may take for instance $f = u + u^2 + u'^2$, or $f = |u| + |u'|$, or $f = |u|u'$. As another example we may consider the system

$$u'' + u = \varepsilon(1-|v|)u'\ , \qquad v'' + 2v = \varepsilon(1-|u|)v'\ ,$$

which has two families of cycles respectively of the forms

$$u = \lambda\omega^{-1}\cos(\omega t + \phi) + O(\varepsilon), \qquad v = O(\varepsilon),$$

$$\omega = \omega(\lambda,\varepsilon) = 1 + O(\varepsilon),$$

and

$$u = O(\varepsilon), \qquad v = \lambda\omega^{-1}\cos(\omega t + \phi) + O(\varepsilon),$$

$$\omega = \omega(\lambda,\varepsilon) = 2^{1/2} + O(\varepsilon),$$

λ, ϕ arbitrary, $|\varepsilon|$ sufficiently small.

As a further example let us consider the third order equation

$$u''' + \sigma^2 u' = \varepsilon f(u,u',u'',\varepsilon),$$

where $f(u,-u',u'',\varepsilon) = -f(u,u',u'',\varepsilon)$. J. K. Hale proved that this equation has a family of cycles of the form

$$u = c_1\sigma^{-1}\cos\sigma t + c_2 + O(\varepsilon),$$

with $u(c_1,c_2,\varepsilon,-t) = u(c_1,c_2,\varepsilon,t)$, where σt can be replaced by $\sigma t + \phi$, c_1, c_2, ϕ arbitrary, $|c_1|$, $|c_2| \le r_2 < R$, and $|\varepsilon|$ sufficiently small. Thus, for $|\varepsilon|$ sufficiently small, the third equation above has a three-parameter family of cycles.

Also, the question of the asymptotic stability of the periodic solutions of system (1), and of the asymptotic orbital stability of cycles of autonomous systems, can be discussed by using essentially the same technique, as shown in papers by Cesari [9], J. K. Hale [20,26-28], R. A. Gambill [14-16], H. R. Bailey and R. A. Gambill [1].

A. Halanay in his book [18] described the method of successive approximations mentioned above. More extensive accounts of the research described in this Part I are in the books by J. K. Hale [29] and by Cesari [10]. Finally, it should be mentioned here that J. Mawhin, by the more general method we shall describe in Part III for boundary value problems not necessarily of the perturbation type, has improved some of the results for problems of the perturbation type which had been obtained by the present approach (see Part III for references to Cesari's extended method and papers by Mawhin).

References for Part I

[1] H. R. Bailey and R. A. Gambill, On stability of periodic solutions of weakly nonlinear differential equations, J. Math. Mech. 6, 1957, 655-668.

[2] C. Banfi, Sulla determinazione delle soluzioni periodiche di equazioni non lineari periodiche, Bollettino Unione Mat. Ital. (4) 1, 1968, 608-619.

[3] C. Banfi, Su un metodo di successive approssimazioni per lo studio delle soluzioni periodiche di sistemi debolmente non lineari, Atti Accad. Sci. Torino 100, 1968, 1065-1066.

References for Part I (Continued)

[4] C. Banfi e G. Casadei, Calcolo di soluzioni periodiche di equazioni differenziali non lineari, Calcolo, vol. 5, suppl. 1, 1968, 1-10.

[5] L. Cesari, Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici, Mem. Accad. Italia (6) 11, 1941, 633-695.

[6] L. Cesari, Existence theorems for periodic solutions of nonlinear Lipschitzian differential systems and fixed point theorems. Contributions to the theory of nonlinear oscillations 5, 1960, 115-172 (Annals of Math. Studies, Princeton, No. 45).

[7] L. Cesari, Second order linear differential systems with periodic L-integrable coefficients (with J. K. Hale). Riv. Mat. Univ. Parma 5, 1954, 55-61.

[8] L. Cesari, A new sufficient condition for periodic solution of weakly nonlinear differential systems (with J. K. Hale). Proc. Am. Math. Soc. 8, 1957, 757-764.

[9] L. Cesari, Boundedness of solutions of linear differential systems with periodic coefficients (with H. R. Bailey), Arch. Ratl. Mech. Anal. 1, 1958, 246-271.

[10] L. Cesari, Asymptotic behavior and stability problems in ordinary differential equations, vii + 271. Ergebn. d. Math., No. 16, Springer Verlag 1959; 2^{d} ed., 1963; 3^{d} ed., 1971. Russian ed., MIR, Moscow 1964.

[11] L. Cesari, Existence theorems for periodic solutions of nonlinear differential systems. Symposium Differential Equations, Mexico City 1959. Boletin Soc. Mat. Mexicana 1960, 24-41.

[12] L. Cesari, Branching of cycles of autonomous nonlinear differential systems. Math. Notae Univ. Litoral, Rosario, 1, 1962, 231-247.

References for Part I (Continued)

[13] L. Cesari, Un nuovo criterio di stabilità per le soluzioni delle equazioni differenziali lineari. Annali Scuola Normale Sup. Pisa (2) 9, 1940, 163-186.

[14] R. A. Gambill, Stability criteria for linear differential systems with periodic coefficients, Riv. Mat. Univ. Parma 5, 1954, 169-181.

[15] R. A. Gambill, Criteria for parametric instability for linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma 6, 1955, 37-43.

[16] R. A. Gambill, A fundamental system of real solutions for linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma 7, 1956, 311-319.

[17] R. A. Gambill and J. K. Hale, Subharmonic and ultraharmonic solutions for weakly nonlinear systems, J. Ratl. Mech. Anal. 5, 1956, 353-398.

[18] A. Halanay, Differential Equations, Academic Press 1966, particularly pp. 308-317.

[19] J. K. Hale, Evaluations concerning products of exponential and periodic functions. Riv. Mat. Un. Parma 5, 1954, 63-81.

[20] J. K. Hale, On boundedness of the solutions of linear differential systems with periodic coefficients, Riv. Mat. Univ. Parma 5, 1954, 137-167.

[21] J. K. Hale, Periodic solutions of nonlinear systems of differential equations, Riv. Mat. Univ. Parma 5, 1955, 281-311.

[22] J. K. Hale, On a class of linear differential equations with periodic coefficients, Illinois J. Mat. 1, 1957, 98-104.

[23] J. K. Hale, Linear systems of first and second order differential equations with periodic coefficients. Illinois J. Math. 2, 1958, 586-591.

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References for Part I (Concluded)

[24] J. K. Hale, Sufficient conditions for the existence of periodic solutions of systems of weakly nonlinear first and second order differential equations. J. Math. Mech. 7, 1958, 163-172.

[25] J. K. Hale, A short proof of a boundedness theorem for linear differential systems with periodic coefficients, Arch. Ratl. Mech. Anal. 2, 1959, 429-434.

[26] J. K. Hale, On the behaovior of the solutions of linear periodic differential systems near resonance points. Contributions to The Theory of Nonlinear Oscillations, vol. V, pp. 55-89, Princeton Univ. Press, 1960.

[27] J. K. Hale, On the stability of periodic solutions of weakly nonlinear periodic and autonomous differential systems, Contributions to the Theory of Nonlinear Oscillations, vol. V, pp. 91-113, Princeton Univ. Press, 1960.

[28] J. K. Hale, On the characteristic exponents of linear periodic differential systems, Bol. Soc, Mat. Mexicana (2) 5, 1960, 58-66.

[29] J. K. Hale, Oscillations in Nonlinear Systems, McGraw-Hill 1963.

[30] C. Imaz, Sobre ecuaciones differenciales lineales periodicas con un parametro pequeno, Bol. Soc. Mat. Mexicana (2) 6, 1961, 19-51.

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II. PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Here again, as in Part I, we describe another aspect of the same process of which we shall see a more general form in Part III for boundary value problems. We shall consider here the question of the possible periodic solutions of nonlinear differential equations of the type

$$u_{xy} = f(x,y,u,u_x,u_y) , \qquad u = (u_1,\dots,u_n) , \tag{1}$$

or of the systems of nonlinear wave equations

$$u_{xx} - u_{yy} = f(x,y,u,u_x,u_y) , \qquad u = (u_1,\dots,u_n) , \tag{2}$$

or of other systems of nonlinear hyperbolic partial differential equations. Here again it was shown by Cesari [1-6] that, by taking into consideration suitably relaxed problems, it is possible to prove, for the relaxed problems, general existence theorems, uniqueness theorems, and theorems of continuous dependence upon the data. The solutions of these relaxed problems are then solutions of the original problems whenever corresponding "bifurcation equations" can be satisfied. J. K. Hale, D. Petrovanu, G. Hecquet, and others have further developed this kind of argument in the present context.

As an example, let us consider the problem of the solutions $u(x,y)$, of system (1), periodic in x of some period L in a strip $-\infty < x < +\infty$, $-a \leq y \leq a$, with $a > 0$ sufficiently small, and f also periodic in x of period L. Cesari [1,5] found that a suitably relaxed problem is of the form

$$u_{xy} = f(x,y,u,u_x,u_y) - m(y) , \quad m(y) = L^{-1}\int_0^L f dx ,$$

and this corresponds to the projection operator P mapping any function $v(x,y)$ periodic in x of period L, into its mean value Pv with respect to x. The solution $u(x,y)$ of the relaxed problem is uniquely determined by Darboux data $u(x,0) = u_0(x)$, $u(0,y) = v_0(y) + u_0(0)$, $v_0(0) = 0$. Criteria are then given in order that, for a given $u_0(x)$, we can determine $v_0(y)$, $-a \leq y \leq a$, for $a > 0$ sufficiently small, in order that the bifurcation equation $m = 0$ be satisfied. One of these criteria was actually derived in [5] from a novel implicit function theorem of the hereditary type based on functional analysis (Cesari [4]).

As another example, let us consider the problem of the solutions $u(x,y)$, $(x,y,)\in E_2$, periodic in x and y of a given period L, of system (1) with f also periodic in x and y of the same period. Cesari [3] found that a suitably relaxed problem is of the form

$$u_{xy} = f(x,y,u,u_x,u_y) - m(y) - n(x) - \mu, \tag{3}$$

$$m(y) = L^{-1}\int_0^L f\,dx, \quad n(y) = L^{-1}\int_0^L f\,dy, \quad \mu = L^{-2}\int_0^L\int_0^L f\,dx\,dy,$$

corresponding to the projection operation P mapping any doubly periodic function $v(x,y)$ into Pv, the sum of the mean values of v with respect to x, with respect to y, and with respect to (x,y).

Let us consider more closely the last mentioned problem. Let $N, N_1, N_2, K, b_1, b_2, M_1, M_2, M_3 \geq 0$ be given constants satisfying

$$M_1 \geq N + 2^{-1}(N_1+N_2)L + 3KL^2, \; M_2 \geq N_1 + 3KL, \; M_3 \geq N_2 + 3KL.$$

and let R denote the set $R = E_2 \times [u,p,q,\epsilon E_n \mid |u| \leq M_1, \; |p| \leq M_2, \; |q| \leq M_3]$. Let $u_o(x)$, $v_o(y)$ be periodic functions of period L, continuous with their first derivatives $u_o'(x)$, $v_o'(y)$, satisfying $v_o(0) = 0$, $|u_o(0)| \leq N$, $|u_o'(x)| \leq N_1$, $|v_o'(y)| \leq N_2$, and let f be continuous in R with $|f(x,y,u,p,q)| \leq K$, and

$$|f(x,y,u,p_1,q_1) - f(x,y,u,p_2,q_2)| \leq b_1|p_1-p_2| + b_2|q_1-q_2|.$$

If, in addition,

$$2Lb_1 < 1, \qquad 2Lb_2 < 1, \tag{4}$$

then Cesari [3] proved, by application of Schauder's fixed point