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GEOMETRY OF HOMOGENEOUS BOUNDED DOMAINS

Coordinatore : Edoardo Vesentini

CENTRO INTERNAZIONALE MATEMATICO ESTIVO
(C.I.M.E.)

S.G. GINDIKIN, I.I. PJATECCKIĬ-ŠAPIRO, E.B. VINBERG

"HOMOGENEOUS KÄHLER MANIFOLDS"

Corso tenuto ad Urbino dal 5 al 13 luglio 1967

HOMOGENEOUS KÄHLER MANIFOLDS [1]

by

S.G. GINDIKIN, I.I. PJATECCKIĬ-ŠAPIRO, E.B. VINBERG

Introduction. Recall of certain results.

1. Definition of homogeneous Kähler manifolds.

Let $h = g + i$ be a positive definite Hermitian differential form on the complex manifold M. Then g is a positive definite symmetric differential form and η is a non-degenerate skew-symmetric differential form of type $(1,1)$, and

$$g(x,y) = \eta\,(Ix,y) \tag{1}$$

where I is the complex structure operator. The complex manifold M with the positive definite hermitian differential form h is called Kählerian if one of the following equivalent conditions is satisfied

(K1) $$d = 0 \;;$$

(K2) The parallel translation with respect to the riemannian metric g preserves the complex structure of the tangent space, i.e. $\nabla I = 0$;

(K3) In local coordinates z^α, $\bar z^\alpha$ the coefficients $h_{\alpha\bar\beta}$ of the form h can be represented in the form

$$h_{\alpha\bar\beta} = \frac{\partial^2 \log\varphi}{\partial z^\alpha\,\partial \bar z^\beta} \tag{2}$$

where φ is a positive real function.

The prof of the equivalence of conditions (K1) - (K3) can be found for example in [13, 27] .

An automorphism of the Kähler manifold M is an invertible

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(1) English Translation by Adam Koranyi.

holomorphic map preserving the form h. We shall denote the group of all automorphisms of the Kähler manifold M by $G(M)$; its connected component by $G^{o}(M)$. We shall also consider the group $G_A(M)$ of all invertible holomorphic transformations of the manifold M and the group $G_R(M)$ of all isometries of M as a riemannian manifold. Then

$$G(M) = G_A(M) \cap G_R(M) .$$

We denote by $G^{o}_A(M)$ and $G^{o}_R(M)$ the connected components of the groups $G_A(M)$ and $G_R(M)$ respectively. In [8, 12] some sufficient conditions are given in order that $G^{o}_R(M) = G^{o}(M)$. We are not going to discuss these conditions here. However the connection between the groups $G_A(M)$ and $G(M)$ will be considered in certain cases.

The Kähler manifold M is called homogeneous if the group $G(M)$ acts transitively on it.

Often the homogeneity of a Kähler manifold is defined by the transitivity of the group $G_A(M)$. From the results of A. Borel - R. Remmert [3] and of Tits [22] it follows that, if a compact Kähler manifold is homogeneous in this sense, then there exists on it a Kählerian structure (compatible with the given complex structure) with respect to which it is a homogeneous in our sense.

In the non-compact case, it is unlikely that the consideration of homogeneous complex manifolds carrying Kählerian structures will lead to a significative classification.

The simplest examples of homogeneous Kähler manifolds are the hermitian space H^n, the complex torus T^n, the complex projective space P^n and the unit disc K in the complex plane. In the following three paragraphs we shall describe three fundamental types of homogeneous Kähler manifolds which have an extremely important significance for the theory.

In the following we shall abbreviate the words "homogeneous Kähler manifolds" by "h.K.m.".

2. Locally flat homogeneous Kähler manifolds.

These are h.K.m.'s which have zero curvature in the Riemannian metric g. They are easy to classify. First of all, every homogeneous locally flat h.K.m. is isomorphic with the hermitian space H^n In fact, by a known theorem of E. Cartan it is isomorphic to an Euclidean space as a riemannian manifold; from the Kähler condition (K2) it follows that the complex structure is invariant under parallel translations.

Any locally flat h.K.m. can be obtained by factoring H^n by some lattice.

The group $G_A^o(M)$ for a locally flat h.K.m. M is the complex hull of the groups $G^o(M)$. A maximal complex subgroup of $G^o(M)$ is the group of parallel translations. It is transitive on M.

3. Simply connected compact homogeneous Kähler manifolds.

These h.K.m.'s have been studied by several authors and have been completely classified (Lichnerowicz [11] , Borel [2] , Wang [26]). We note that Wang [26] found all simply connected complex homogeneous manifolds. Some of these do not admit any Kählerian structure.

We formulate the fundamental result concerning this type of h.K.m. . **Let** M be a simply connected compact h.K.m. . Then the group $G^o(M)$ is a compact semi-simple Lie group with trivial center, its isotropy subgroup is connected and is the centralizer of a torus. Conversely, if G is a connected compact Lie group and K is the centralizer of a torus in G , then there exists an invariant Kähler structure on the homogeneous space G/K . Every complex Lie group has only finitely many sub-groups (up to conjugation) that are centralizers of tori.

They can all be easily found.

Every simply connected compact h.K.m. M can be realized as an algebraic manifold in a complex projective space P^n in such a way that the automorphisms of M will be the restrictions of unitary projective transformations. However the Kähler structure of M will in general be different from the Kähler structure induced by P^n.

As in the locally flat case, the group $G_A^o(M)$ of the simply connected compact h.K.m. M is the complex hull of $G^o(M)$. However in this case $G^o(M)$ has no non-trivial complex subgroups. The isotropy subgroup K_A of the group $G_A^o(M)$ is connected and contains a Borel subgroup (the group K_A is <u>not</u> the complex hull of the group K).

Let us look at a typical example.

Let G be the group of $n \times n$ unitary matrices. Let $K = K(n_1, \ldots, n_s)$, $\sum n_i = n$ be the subgroups consisting of all diagonal block matrices of order $n_1, \ldots, n_s$. We call an $(n_1, \ldots, n_s)$ flag a sequence of subspaces of the hermitian space H^n of dimensions n, $n_1 + n_2, \ldots, n_s + \ldots + n_{s-1}$, contained successively in each other. The homogeneous manifold G/K can be realized as the manifold of $(n_1, \ldots, n_s)$ flags.

In a natural way it is contained in a complex projective space ; the Kähler structure induced by this inclusion is invariant under the group G .

We should mention that the group G in this example acts non effectively on $G/_K$. The kernel of the action is the center of the group G, which is contained in K . This is in agreement with the general theory, since the automorphisms group of a simply connected compact h.K.m. always has a trivial center as we remarked before.

The group G_A of all non-singular complex $n \times n$ matrices is the complex hull of the group G and acts analytically on $G/_K$, but it does not preserve the Kähler structure. The isotropy subgroup of G_A is the the group $K_A = K_A(n_1, \ldots, n_s)$ which consists of all triangular block matrices with blocks of order $n_1, \dots, n_s$ on the diagonal.

This is how one describes (up to the choice of the Kählerian structure) all simply connected complex h.K. m. which are connected with the unitary group. For the other compact Lie groups there is an analogous construction.

Matsushima [14] proved that every compact h.K.m. is a direct product of a simply connected compact h.K.m. and a complex torus.

4. Homogeneous bounded domains.

Let D be a bounded domain in the n-dimensional complex space $\mathbb{C}^n$. The Bergman metric [1, 5, 27] defines in D a canonical Kählerian structure. This structure is invariant under all analytic automorphisms of the domain D, that is, now $G_A(D) = G(D)$.

The domain D is said to be homogeneous if the group $G_A(D)$ is transitive on it.

In the case of a homogeneous domains the coefficients of the Bergman metric can be found on the basis of (2) where for φ one has to take the density of the invariant measure. Beside the canonical Kählerian structure there may exist other Kählerian structures in a homogeneous domain which are invariant with respect to $G_A(D)$, or with respect to some transitive subgroup of it. Differently from the case of the other fundamental types of h.K.m., for the bounded domains D, the group $G_A(D)$ does not contains any non-trivials complex subgroup.

In the following we shall abbreviate the word "homogeneous bounded domains" as "h.b.d.".

In $\mathbb{C}^1$, up to analytic isomorphisms, there is only one h.b.d.: the unit disc $\{|z| < 1\}$. In $\mathbb{C}^2$ there exist two non-isomorphic h.b.d.'s: the complex ball

$$\{|z_1|^2 + |z_2|^2 < 1\}$$

and the bi-cylinder

$$\{|z_1| < 1, |z_2| < 1\}.$$

The non-isomorphy of these domains was proved by Poincaré [15]. The non-existence of other h.b.d. in $\mathbb{C}^2$ was shown by E. Cartan [4]. He also found all h.b.d.'s in $\mathbb{C}^3$.

The domain $D \subset \mathbb{C}^4$ is called <u>symmetric</u> if for every point $z \in D$ there exists an involutive analytic automorphism b_z of D for which z is a unique fixed point.

Every symmetric bounded domain is homogeneous and is a symmetric space.

Using the classification of symmetric spaces, E. Cartan enumerated all bounded symmetric domains [4]. In the same work he established that for $n \leqslant 3$ all h.b.d.'s are symmetric. In connection with this he posed the problem: are all h.b.d.'s symmetric? And if not, how can one construct them?

A. Borel [2] and Koszul [9] showed that if a h.b.d. there is acted upon by semi-simple group of analytic automorphisms, then their domain is symmetric. The same result, with still weaker hypotheses, was proved by Hano [7].

In [16] Pjatecckii - Šapiro obtained a negative answer to the first part of E. Cartan's problem. He constructed an example of a non-symmetric h.b.d. in $\mathbb{C}^4$. (We shall describe it in § 6.)

It turned out later that the symmetric domains are in a certain sense exceptional among the h.b.d.'s in $\mathbb{C}^n$, while for every n there are only finitely many bounded symmetric domains in $\mathbb{C}^n$. It is interesting that the non-symmetric h.b.d.'s arise naturally in connection with the study of homogeneous fiberings of symmetric domains [17, 21] .

In [18, 19, 20] Pjatecckii-Šapiro studied in detail those h.b.d.'s which admit a transitive solvable group of automorphisms acting without fixed points. He proved that every such h.b.d. is isomorphic with a non-bounded homogeneous domain, which is homogeneous under a group of affine transformations (a description of these domains, so-called Siegel domains of type I and II, will be given in the first part of these lectures) ;

In the joint work [25] by Vinberg, Gindikin and Pjatecckii-Šapiro the same result was obtained without any restrictive hypothesis. It turned out a posteriori that the condition imposed by Pjatecckii-Šapiro is not really a restriction. For every h.b.d. D (with any homogeneous Kählerian structure) in the group $G^o(D)$ there is a transitive splittable solvable subgroup T(D)) acting on D without fixed points. There exists a realization of D as a convex unbounded domain such that the elements of the group T(D) are affine transformations. The group $G^o(D)$ has no center. The isotropy subgroup is a maximal compact subgroup in $G^o(D)$.

All these results were obtained in [25] . In these lectures we prove some theorems about h.K.m.'s from which the result of Pjatecckii-Šapiro follows under the hypothesis that the domain D admits a transitive splittable solvable group of automorphisms.

5. The structure of arbitrary homogeneous Kähler manifolds.

Every h.K.m. which has a transitive semi-simple group of automorphisms admits a holomorphic fibering with a simply connected h.K.m. as its fiber, the base of which is analitycally isomorphic with a symmetric bounded domain (Borel [2] , Matsuhima [14]). Every h.K.m. which has a transitive reductive group of automorphisms decomposes into the direct product of an h.K.m. admitting a transitive semi-simple group of automorphisms and of a locally flat h.K.m. (Matsushima [14]) .

These theorems and several other results, some of which will be discussed below, gave us the basis for the following conjecture.

Fundamental conjecture. Every homogeneous Kähler manifold admits a holomorphic fibering, the base of which is analytically isomorphic with a homogeneous bounded domain, and the fiber, with the induced Kähler structure, is isomorphic with the direct product of a locally flat h.K.m. and a simply connected compact h.K.m. .

Besides the cases mentioned above (results of A. Borel and Matsushima) this conjecture is essentially proved, even though this is not explicitely mentioned, in our article [25] for h.K.m. 's, which admit a transitive group of automorphisms on which the pre-image of the differential form η = Im h (cf. § 1) is the differential of some left-invariant form. In this case there is no locally flat factor in the fiber.

A considerable part of these lectures will deal with the proof of the fundamental conjecture for Kähler manifolds which admit a transitive splittable solvable group of automorphisms.

This result is due to Vinberg and Gindikin.

Let us make some remarks in connection with the fundamental conjecture. The fibering about which we have spoken is unique, since its fibers can by characterized as the maximal sets on which all bounded holomorphic functions are constant. Therefore it is preserved by all analytic automorphisms of the manifold. Furthermore the base of the fibering, being a h.b.d. , is homeomorphic with an affine space. Consequently this fibering is topologically trivial. Its structure group is the group of invertible holomorphic map of the fiber, and is a complex Lie group (cf. §§ 2 and 3) . According to a theorem of Grauert [6] such holomorphic fiberings are trivial.

Therefore if the fundamental conjecture is true, then every h.K.m. is, as a complex manifold, isomorphic with the direct product of h.K.m's of the three fundamental types described in §§ 2 - 4 .

PART I - Siegel domains .

1. Siegel domains of type I .

We have already spoken in the introduction about the important role played in the theory of h.b.d.'s by their affine homogeneous realizations. In the case of symmetric domains we usually consider their realization as "disc".

In these realizations the isotropy group of some point of the domain consists of linear transformations. Here we shall consider other realizations of the type of the "upper half plane" in which there is a transitive group of affine transformations (this group can be interpreted as the isotropy group of a point of the boundary of the domain). In the course of this, we shall consider certain special classes of affine homogeneous domains : the Siegel domains of type I and II . In this paragraph we shall talk about the following simplest generalization of the upper half plane to the case of several complex variables.

Let V be an open convex cone in the n dimensional real space $\mathbb{R}^n$ (i.e. if $x, y \in V$, then $\lambda x + \mu y \in V$ for $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu \neq 0$) not containing any straightline .

The domain in $\mathbb{C}^n$

(1) $$D(V) = \mathbb{R}^n + i\, V$$

is called a Siegel domain of type I associated with the cone $V^{(1)}$.

Proposition I. Every Siegel domain of type I is isomorphic with a bounded domain.

Proof. The convex cone V, containing no straight line, is contained in some n - sided angle. Making a linear transformation of $\mathbb{R}^n$, this angle can be transformed into the positive octant of $\mathbb{R}^n$, V' : $y_k > 0$ $(k = 1, \dots, n)$. If this transformation is continued by the same formulas to $\mathbb{C}^n$, then the domain $D(V)$ becomes a part of the domain $D(V')$. The domain $D(V')$, being a direct product of upper half planes, is analytically isomorphic with the n-dimensional circular poly-cylinder

$$\left\{ |z_k| < 1 , \quad k = 1, \dots, n \right\} .$$

That is the domain $D(V)$ can be mapped into a subset of this poly-cylinder.

We introduce an auxiliary notion. We call skeleton of the domain $D \subset \mathbb{C}^n$ a set Ω_D such that :

α) every function $f(z)$ which is holomorphic on $\bar{D}$ (the closure of D) and assuming its maximum modulus in $\bar{D}$, reaches its maximum modulus in some point of Ω_D ;

β) for every point $z_0 \in \Omega_D$ there exists a function holomorphic in $\bar{D}$ whose modulus assumes its maximum in the point z_0 and only there.

It is clear that the skeleton Ω_D is uniquely defined, if it exists, and that it is preserved by all automorphisms of the domain D which are holomorphic on $\bar{D}$.

Lemma 1. The skeleton of the Siegel domain of type I $D(V)$ is the set $\mathbb{R}^n$

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(1) For Siegel domain of type I a more widespread name is "radiated tube domains".

Proof: α). Let the maximum modulus of the function $f(z)$ holomorphic in $\bar{D}$ be assumed at the point $z_0 \in D$, $\operatorname{Im} z_0 = 0$.

We may assume that $\operatorname{Re} z_0 = 0$. The function of one variable

$$\varphi(\lambda) = f(\lambda z_0)$$

will be holomorphic in the half plane $\{\operatorname{Re} \lambda > 0\}$ and its modulus reaches its maximum for $\lambda = 1$. But then it assumes its maximum also for $\lambda = 0$.

β) From the considerations of the proof of proposition 1, it is clear that, without restriction of generality, we may assume that the domain $D(V)$ is contained in a direct product of upper half planes

$$\left\{\operatorname{Im} z_k > 0,\ k = 1, \dots, n\right\} .$$

Then for the point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$, the function

$$f(z) = \frac{1}{(z_1 - x_1^0 + i) \dots (z_n - x_n^0 + i)}$$

will satisfy the condition β) in the definition of the skeleton.

(The point x_0 will be the unique maximum of $|f(z)|$ in the domain $\left\{\operatorname{Im} z_k \geqslant 0,\ k = 1, \dots, n\right\}$ and therefore also in $D(V)$).

Let V be a cone in $\mathbb{R}^n$ having the properties mentioned above.

We denote by $G(V)$ the group of non singular linear transformations of $\mathbb{R}^n$ which preserve V. A cone V is called <u>homogeneous</u> if $G(V)$ acts transitively on V. Analogously to Siegel domain of type I $D \subset \mathbb{C}^n$, we denote by $G_a(D)$ the group of affine transformations of $\mathbb{C}^n$ which preserve D. If the group $G_a(D)$ acts transitively on D, then we shall call D a <u>homogeneous Siegel domain of type</u> I. Here we shall not explain more precisely the term "affine homogeneous Siegel domain" since in the case of Siegel domains one always talks about homogeneity with respect to an affine group.

Proposition 2. The Siegel domain of type I $D(V)$ is homogeneous if, and only if, the cone V is homogeneous.

Proof 1. If V is a homogeneous cone and $G(V)$ is a transitive group of automorphisms on it, then the maps of $\mathbb{C}^n$ of the form

$$(2) \qquad z \longrightarrow A z + a ,$$

where $A \in G(V)$ (more exactly, we have to take the complex continuation of the linear transformations of $\mathbb{R}^n$), $a \in \mathbb{R}^n$, form a transitive group in $D(V)$.

2. We show the converse. Let $D(V) = D$ be a homogeneous Siegel domain of type I, and let (2) be an affine transformation preserving $D(V)$ (A is a non-singular complex linear transformation, $a \in \mathbb{C}^n$). Under such a map the skeleton Ω_D must be preserved. This follows from the general fact that the skeleton is preserved by maps which are analytic on the closure of the domains. However in our case it is also enough to mention that the skeleton Ω_D is a maximal flat component of the boundary of D containing 0. Since the skeleton $\Omega_D = \mathbb{R}^n$ is preserved, the linear transformation A and the vector a must be real. Then for our automorphisms, $y = \operatorname{Im} z$ is acted upon by the transformation A and the cone V must be preserved. From this it follows that the linear transformations A entering in (2) form a transitive group of linear automorphisms of the cone V. In order to construct examples of homogeneous Siegel domains of type I, it is sufficient to construct examples of homogeneous convex cones not containing straight lines.

Example 1. Consider the cone V of symmetric positive definite matrices g of order ℓ. This is a convex cone, containing no straight lines, in the space $\mathbb{R}^n$, $n = \frac{\ell(\ell+1)}{2}$, of symmetric matrices of order ℓ . The automorphisms of the cone are the mappings

(3) $$y \to g\, y\, g' ,$$

where g is a non-singular matrix of order ℓ , and g' is its transposed. This is the formula for the change of the matrix of a quadratic form under a change of variables. The transitivity of the group $G(V)$ follows form the possibility of reducing every positive definite quadratic form to a sum of squares. We mention that the transitivity is preserved if we restrict ourselves in (3) to triangular (for example upper triangular) matrices with positive diagonal elements .

The corresponding Siegel domain of type I $D(V)$ (in this case it is usually called "Siegel upper half plane") consists of the complex symmetric matrices of order ℓ with positive definite imaginary part.

Example 2. As another example we consider the cone V of complex hermitian positive definite matrices of order ℓ , considered as a cone in the real space $\mathbb{R}^n$, $n = \ell^2$ of hermitian matrices. In it there acts transitively the group of non-singular complex matrices g :

(4) $$y \to g\, y\, g^* ,$$

where $g^* = \bar{g}'$; here the group of triangular matrices with real positive diagonal elements acts on V transitively without fixed points.

The corresponding Siegel domain of type I can be realized as the set of these complex matrices z of order ℓ , for which the hermitian matrix $\frac{1}{2i}(z - z^*)$ is positive definite.

The Siegel domains of example 1, 2, are symmetric. In both cases, the symmetry at the point $z = i\,E$ (E is the identity matrix) is given by :

(5) $$z \to -z^{-1}$$

It is clear that it is sufficient to give the involution at one point.

It turns out that a Siegel domain of type I is symmetric if, and only if, the cone V is self-adjoint with respect to some scalar product (The adjoint cone V^* consists of these x for which the inner product $\langle x , y \rangle > 0$ for all $y \in \bar{V}$, $y \neq 0$). We are not going to prove this result here.

Example 3. In order to construct non-symmetric homogeneous Siegel domains of type I, one has to construct homogeneous non-self-adjoint cones (with respect to any scalar product). Such cones appear first in $\mathbb{R}^5$. Consider the cone in $\mathbb{R}^5$:

$$(6) \qquad \left\{ \begin{array}{l} y_{11} \, y_{33} - y_{13}^2 > 0 \\ y_{22} \, y_{33} - y_{23}^2 > 0 \\ \qquad\qquad y_{33} > 0 \, . \end{array} \right.$$

Its adjoint cone is the cone of symmetric positive definite matrices of the form

$$\begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{22} & 0 \\ y_{13} & 0 & y_{33} \end{pmatrix} .$$

This cone is not linearly equivalent with the cone (6).

Correspondingly there exist homogeneous non-symmetric Siegel domains of type I in $\mathbb{C}^n$ for $n \geqslant 5$. Let us recall (cf. also the following paragraph) that non-symmetric h.b.d.'s exist in $\mathbb{C}^n$ for $n \geqslant 4$. Let us mention also that there exist an analytic continuum of non-isomorphic Siegel domains of type I in $\mathbb{C}^n$, for $n \geqslant 11$ (in the class of all h.b.d.'s there is a continuum of non isomorphic ones for $n \geqslant 7$).

2. Siegel domains of type II.

From the concluding remarks of the previous paragraph one can infer that not all h.b.d.'s are isomorphic with Siegel domains of type I. One can get to the same conclusion from simplex considerations too. For the complex ball

$$|z_1|^2 + \dots + |z_n|^2 < 1$$

with $n \geq 2$, there exists no realization as a Siegel domain of type I. This fact will in full be a consequence of the results of the following part, but let us show right now that the complex ball cannot be mapped onto a Siegel domain of type I by a mapping which is holomorphic on the closed ball. For the proof it is sufficient to remark that the skeleton of a Siegel domain of type I has real dimension $n = \dim_{\mathbb{C}} D$, while the skeleton of the ball coincides with its boundary, i.e. has real dimension $2n-1$. This follows from the fact that the modulus of the function $\dfrac{1}{z_1 - \frac{1}{2}}$ reaches its maximum only at the point $(1, 0, \dots, 0)$ in the closed ball, and the group of unitary linear transformations acts transitively on the boundary of the ball. The ball can be mapped onto an affine homogeneous domain by setting

$$z_1 = \frac{z-i}{z+i}, \quad z_2 = \frac{u_1\sqrt{2}}{z+i}, \dots, z_n = \frac{u_{n-1}\sqrt{2}}{z+i} .$$

We obtain as image the domain

$$\text{(7)} \qquad \operatorname{Im} z - |u_1|^2 - \dots - |u_{n-1}|^2 > 0 .$$

We describe now a transitive group of affine transformations of the domain (7). We consider the maps

$$\text{(8)} \qquad \begin{aligned} z &\to z + a + 2i \sum u_k \bar{c}_k + i \sum c_k{}^2 , \\ u &\to u + c \end{aligned}$$

where $a \in \mathbb{R}$, $c \in \mathbb{C}^{n-1}$

It easy to check that the domain (7) is preserved by the mappings (8). Besides we have the automorphisms

$$(9) \qquad \begin{aligned} z &\to \lambda^2 z , \\ u &\to \lambda u \end{aligned} \qquad (\lambda > 0) .$$

The mappings (8), (9) generate a transitive group. In fact any point (z, u) satisfying (7) can be mapped by (8) onto a point $(iy, 0)$ $(y > 0)$, and this point can be mapped by (9) onto $(i, 0)$. For the proof of the transitivity of the group of automorphisms it is enough to prove that an arbitrary point of the domain can be carried onto some given point.

The construction (7) admits the following generalization.

Let V be a convex cone in R^n not containing straight lines. The map $F: \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ will be called a V - <u>hermitian form</u> if

(10) $F(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 F(u_1, v) + \lambda_2 F(u_2, v) \quad (\lambda_1, \lambda_2 \in \mathbb{C}; u_1, u_2, v \in \mathbb{C}^m)$,

(11) $F(u, v) = \overline{F(v, u)}$,

(12) $F(u, u) \in \bar{V}$, where $\bar{V}$ is the closure of the cone V ,

(13) $F(u, u) = 0$ only if $u = 0$.

In the case where V is the positive half line, a V-hermitian form is a usual positive definite hermitian form .

<u>Siegel domain of type II</u> $D(V, F)$ associated to the cone V and to the V-hermitian form F is the domain in $\mathbb{C}^{n+m}$ consisting of the points (z, u), $z \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ satisfying the condition

$$(14) \qquad \operatorname{Im} z - F(u, u) \in V .$$

For $n = 1$ we obtain the domain (7) . The Siegel domains of type I can be considered as special cases of Siegel domains of type II $(m = 0)$.

First of all we prove following

Proposition 3 . Every Siegel domain of type II, $D(V,F)$, is analytically isomorphic will some bounded domain .

Proof. Since the V-hermitian form F becomes a V'-hermitian form if we change the cone V to a cone $V' \supset V$, we can include the domain $D(V,F)$ in a domain $D(V',F)$, where V' is an octant. So we can restrict ourselves to the case where V is such an octant, and we can assume that it is the positive octant of $\mathbb{R}^n$ (by making a linear transformation if necessary). In this case all components of F : F_1, ..., F_n will be non-negative definite hermitian forms.

We represent each of the forms F_k as a sum of squares of moduli of linear forms

$$F_k(u,u) = \sum_j \left| L_{jk}(u) \right|^2 . \tag{15}$$

From the set of all forms L_{jk} we choose a maximal set S of linearly independent forms is equal to m since by (13) the forms L_{jk} have a unique common zero. In the sums (15) we replace by 0 all forms which do not occur in the system S .

We denote the resulting hermitian forms by $\tilde{F}_k$. The domain $D(V,\tilde{F})$ contains the domain $D(V,F)$. Choosing the forms in S as new variables in $\mathbb{C}^m$ we obtain that the domain $D(V,F)$ in these variables is the direct product of n domains of the form (7) , that is analytically isomorphic with the direct product of n balls.

The proof is complete.

Now we consider the question of the automorphisms of a Siegel domain of type II. A Siegel domain of type II $D \subset \mathbb{C}^{n+m}$ is called homogeneous if the group $G_a(D)$ of these affine transformations of $\mathbb{C}^{n+m}$ which preserve D acts transitively on D In a Siegel domain of type II we have always the analogues of the transformations (8) :

$$(16)\qquad \begin{cases} z \to z + a + 2i\ F(u, c) + i\,(F(c, c)\ , \\ u \to u + c \qquad\qquad (a \in \mathbb{R}^n\ ,\ c \in C^m)\ . \end{cases}$$

Before determining the general form of affine automorphisms , we study the skeleton of Siegel domains of type II .

Lemma 2. The skeleton Ω_D of the Siegel domain of type II $D = D(V, F)$ consists of those points (z, u) for which $\mathrm{Im}\, z = F(u, u)$.

Proof. Let $f(z)$ be a holomorphic function in $\bar{D}$ whose modulus assumes its maximum at the point $(z_0, u_0) \in \bar{D}$, $\mathrm{Im}\ z_0 \neq F(u_0, u_0)$. Using the mappings (16) , we can assume that $u_0 = 0$, $\mathrm{Re}\ z_0 = 0$, i.e. $z_0 = i\, y_0$, $y_0 \in \bar{V}$, $y_0 \neq 0$. Then , just as in the proof of Lemma 1, the function $\varphi(\lambda) = f(\lambda\ z_0, 0)$ will be holomorphic in the upper half-plane, and its modulus assume its maximum in the point $\lambda = 1$.

Using the mapping (16) it is enough to prove property (β) for $u_0 = 0$. In that case, if $(z_0, 0) \in \overline{D(V, F)}$, then $z_0 \in \overline{D(V)}$, and the function constructed in the proof of lemma 1 will satisfy the required conditions (it is essential that when $(z, u) \in \overline{D(V, F)}$, then $z \in \overline{D(V)}$, and $\mathrm{Im}\ z \neq 0$ for $u \neq 0$ by (13)) .

Now we shall study the general form of the automorphisms if a Siegel domain of type II.

Proposition 4. Every affine automorphism preserving the Siegel domain $D(V, F)$ of type II is of the form :

$$(17)\qquad \begin{cases} z \to A\ z + a + 2i\ F(B\ u, c) + i\ F(c, c) \\ u \to B\ u + c \end{cases}$$

where $a \in \mathbb{R}^n$, $c \in \mathbb{C}^m$, A is an automorphism of the cone V, B is a linear transformation of $\mathbb{C}^m$ such that

$$(18)\qquad AF(u, u) = F(B\, u\ ,\ B\ u)\ .$$

<u>Proof.</u> Every mapping (17) is composed of a map (16) and of a map

$$(19) \qquad \begin{cases} z \to A\ z \\ u \to B\ u \end{cases}$$

where A and B satisfy condition (18). It is clear that under this condition, (19) is an automorphism of the domain $D(V, F)$.

Suppose that we have an affine automorphism of the domain $D(V, F)$

$$(20) \qquad \begin{cases} z \to L_{11}\ z + L_{12}\ u + b_1\ , \\ u \to L_{21}\ z + L_{22}\ u + b_2\ . \end{cases}$$

Combining (20) with a mapping (16), we can arrange that $b_2=0$, $\operatorname{Re}\ b_1 = 0$. We shall consider mappings of this form.

Furthermore the map (20) must preserve the skeleton Ω_D.

Therefore the point $(0,0)$ must be transformed into a point of the skeleton, i.e. $\operatorname{Im}\ z = F(u,u)$. It follows that $b_1 = 0$, since $(0,0) \to (b_1, 0)$, $\operatorname{Re} b_1 = 0$. So we may assume that in (20) $b_1 = b_2 = 0$.

Consider the points $(x, 0)$, $x \in \mathbb{R}^n$. They belong to Ω_D, and therefore their image belongs to Ω_D, i.e.

$$\operatorname{Im}\ L_{11} = F(L_{21}\ x,\ L_{21}\ x)\ .$$

Since the left-hand side is a linear form, and the right-hand side is a quadratic form, we have

$$\operatorname{Im}\ L_{11} = 0\ , \quad L_{21} = 0\ .$$

Consider the points (iy, u), $y = F(u,u)$. This images must belong to the skeleton; i.e.

$$(21) \qquad L_{11}\ y + \operatorname{Im} L_{12}\ u = F(L_{22}\ u,\ L_{22}\ u)\ .$$

Whence $\operatorname{Im} L_{12} e^{i} u$ is independent of , i.e. $L_{12} u = 0$, and , since this is true for every u, then

$$L_{12} = 0 .$$

By (21) , furthermore

$$L_{11} y = L_{11} F(u,u) = F(L_{22} u, L_{22} u) ,$$

i.e. our map has the form (19) , and A, B satisfy (18) . The proof is complete.

The V-hermitian form F is called homogeneous, if there exists a transitive group G of automorphisms of V such that for every $g \in G$ there exists a linear transformation $\tilde{g}$ of the space $\mathbb{C}^m$ such that

$$g\, F(u,v) = F(\tilde{g}\, u, \tilde{g}\, v) . \tag{22}$$

Corollary 1. The Siegel domain of type II D(V, F) is homogeneous if and only if V is a homogeneous cone and the V - hermitian form F is homogeneous .

For this it is enough to remark that, by a map (16) the point $(z,u) \in D(V,F)$ can be transformed into a point $(i\ y, 0), y \in V$ by proposition 4 . These points must be transformable into each other by maps of the form (19) .

Proposition 4 has the following generalization :

Proposition 5. Every non singular affine transformation mapping the Siegel domain of type II D(V, F) out a Siegel domain of type II $D(V_1, F_1) \subset \mathbb{C}^{n_1 + m_1}$ is of the form

$$\begin{cases} z \to Az + a + 2i\ F_1(Bu, c) + i\ F_1(c,c) , \\ u \to Bu + c, \end{cases}$$

where $a \in \mathbb{R}^{n_1}$, $c \in \mathbb{C}^{m_1}$, A is a linear transformation of the cone

V onto V_1 , B is a linear transformation of $\mathbb{C}^m$ into $\mathbb{C}^{m_1}$ such that

$$A(F(u,u)) = F_1(Bu, Bu) \; .$$

The proof is analogous to the proof of proposition 4.

Corollary. The Siegel domains of type II $D(V,F) \subset \mathbb{C}^{n+m}$ and $D(V_1, F_1) \subset \mathbb{C}^{n_1+m_1}$ are affine equivalent if, and only if, $n = n_1$, $m = m_1$ and there exist isomorphic linear transformations $A : \mathbb{R}^n \to \mathbb{R}^{n_1}$, $B : \mathbb{C}^m \to \mathbb{C}^{m_1}$ such that

$$A\,V = V_1 \; ,$$
$$A\,F(u,u) = F_1(Bu, Bu) \; .$$

The study of homogeneous Siegel domains of type II is reduced to the study of homogeneous V - hermitian forms for homogeneous cones V . The classification of these forms up to linear equivalence for concrete cones is an interesting problem of linear algebra. Let us consider the Siegel domains associated with the cones of example 1 and 2.

Example 4. Let V be the cone of symmetric positive definite matrices of order ℓ . It can always be assumed (cf. the following paragraph) that the map (3), where g is upper triangular matrix with positive diagonal elements, can be continued to $\mathbb{C}^m$, in the sense of (22) . Let us first consider the case where $\mathbb{C}^m$ is a space of rectangular $\ell \times q$ matrices u .

We set

$$F(u,v) = \frac{1}{2}\,(u\,v^* + \bar{v}\,u') \; . \tag{23}$$

It is clear that F is a V-hermitian form. Let us show that is homogeneous. We consider the maps of the cone V :

$$y \to g(y) = t\, g\, t$$

where t is an upper triangular matrix with positive diagonal elements. For $u \in \mathbb{C}^m$ we set

$$\tilde{g}(u) = t\,u\,. \tag{24}$$

For these maps conditions (22) is satisfied.

We obtain other examples of homogeneous V-hermitian forms if we restrict the form (23) to subspaces $\mathbb{C}^{m_0}$ of the space $\mathbb{C}^m$ which are invariant under left multiplication by upper-triangular matrices t (mappings (24)). For this it is necessary that the rows $u_1, \dots, u_\ell$ of the matrix u belong to subspaces $\mathbb{C}^{q_1}, \dots, \mathbb{C}^{q}$ of some chain of subspaces $\mathbb{C}^{q_1} > \dots > \mathbb{C}^{q}$ of the space $\mathbb{C}^q$. It is clear that one can choose a basis in $\mathbb{C}^q$ so that the subspace $\mathbb{C}^{m_0}$ consists of step matrices of some type. (The first elements in each row vanish, and the number of these elements does not decrease when we go from one row to the following).

It turns out that the domains $D(V,F)$ associated to the form (23) are non-symmetric unless $D(V,F)$ is a Siegel domain of type I. We prove this in the simplest case.

Let $\ell = 2$, $q = 1$, let u be matrices with the second row equal to zero. Then we obtain the domain in $\mathbb{C}^4$ given by the following conditions :

$$\begin{vmatrix} y_{11} - |u|^2 & y_{12} \\ y_{12} & y_{22} \end{vmatrix} > 0,\; y_{11} - |u|^2 > 0 \quad (y_{ij} = \operatorname{Im} z_{ij}). \tag{25}$$

This is the first example of a non-symmetric homogeneous bounded domain constructed by I. I. Pjatecckii-Šapiro in [16]. We give a proof that it is not symmetric.

First of all we prove the following lemma :

<u>Lemma</u> 3. The symmetry of the Siegel domain of type II $D(V,F)$

at the point $(z_0, 0)$, if it exists, is of the form

$$(z,u) \longrightarrow (\varphi(z), \psi(z)u)$$

where $z \longrightarrow \varphi(z)$ is the symmetry of the Siegel domain of type I $D(V)$ at the point z_0, and $\psi(z)$ is a linear transformation of $\mathbb{C}^m$ depending analytically on z.

Proof. We mention first that the symmetry is unique at every point (it must be the reflection in the geodesies with respect to the Bergman metric). Let the symmetry at the point $(z_0, 0)$ be

$$(26) \qquad (z,u) \to (\varphi(z,u), \psi(z,u)) .$$

It must commute with every automorphism of $D(V,F)$ which preserves the point $(z_0, 0)$ (because of uniqueness), in particular with

$$(z,u) \to (s, e^{i\theta} u) .$$

Hence

$$\varphi(z, e^{i\theta} u) = \varphi(z,u) ,$$

$$\psi(z, e^{i\theta} u) = e^{i\theta} \psi(z,u) .$$

Because of the analyticity of φ and ψ in a neighborhood of 0 with respect to u we obtain that the symmetry has the form (26). Setting $u = 0$, we obtain that $z \longrightarrow \varphi(z)$ is the symmetry of the domain $D(V)$ at the point z_0.

Lemma 4. The domain (25) is non-symmetric.

Proof. The symmetry at the point $(z = iE, u = 0)$ must be of the form

$$(27) \qquad (z, u) \to (-z^{-1}, \psi(z)u) .$$

Under an analytic automorphism a point of the skeleton Ω_D must go into another point of the skeleton or to infinity.

However under the map (27) the point of the skeleton $z = \begin{pmatrix} i & 1 \\ 1 & 1 \end{pmatrix}$, $u = 1$ goes into the point $z = \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ -1-i & -1+i \end{pmatrix}$ which does not belong to the skeleton. Therefore there exists no symmetry at the point $(i\ E, 0)$.

<u>Remark.</u> It would be possible to compute the volume element for the Bergman metric of the domain (26), and check that it is not invariant under maps of the form (27).

<u>Example</u> 5. Let V be the cone of hermitian positive definite matrices of order ℓ. We realize the space $\mathbb{C}^m$ as the space of pairs of complex rectangular matrices $u^{(1)}$ of type $\ell \times q$ and $u^{(2)}$ of type $(\ell \times r)$. We set

$$F(u, v) = u^{(1)}\ v^{(1)*} + \overline{v^{(2)}}\ u^{(2)} \qquad (28)$$

Let t be an upper triangular (complex) matrix of order ℓ. To the automorphisms of the cone V

$$v \rightarrow g(y) = t\ y\ t^*$$

we make correspond the map of $\mathbb{C}^m$

$$\tilde{g}(u^{(1)}, u^{(2)}) = (t\ u^{(1)}, \bar{t}\ u^{(2)}) \qquad (29)$$

Condition (22) is satisfied. The corresponding domains are symmetric if one of the number q, r is equal to zero.

We denote by u_k the pair $(u_k^{(1)}, u_k^{(2)})$ consisting of the k-th rows of the matrices $u^{(1)}, u^{(2)}$; $u_k \in \mathbb{C}^{q+r}$. If we choose a chain of subspaces $\mathbb{C}^{s_1} \supset \ldots \supset \mathbb{C}^{s}$ in $\mathbb{C}^{q+r}$, and consider the space $\mathbb{C}^{m_0}$ of pairs $u = (u^{(1)}, u^{(2)})$ for which $u_k \in \mathbb{C}^{s_k}$ $(k = 1, \ldots, \ell)$, then we obtain a subspace which is invariant under the maps (29). The restriction of the form (28) to $\mathbb{C}^{m_0}$ gives a homogeneous V - hermitian form. Among the forms so obtained there are families of non-equivalent forms depending on certain parameters. As a result, by proposition 5 we obtain a continuum of affinely non-isomorphic homogeneous Siegel domains of type II. By the results of the next part, they are also analytically non - isomorphic.

The simplest continuous family of non isomorphic domains is obtained for $\ell = 2$, $q = r = 1$, $s_1 = 2$, $s_2 = 1$. In this way we get a family of domains in $\mathbb{C}^7$ (cf. Part II). Besides this family, in $\mathbb{C}^7$ there are only finitely many analitically non isomorphic h.b.d.'s'..

We state now one of the fundamental theorems of the theory of h.b.d.'s in $\mathbb{C}^m$.

Theorem. Every homogeneous bounded domain in $\mathbb{C}^n$ is analytically isomorphic with a homogeneous Siegel domain of type II.

This theorem in its final form was proved by E.B. Vinberg, S.G. Gindikin and I.I. Pjatecckii-Šapiro [25].

In these lectures we will give a proof of it under certain hypotheses concerning the group of automorphisms of the domain.

3. The structure of the group of affine automorphisms of a homogeneous Siegel domain of type II.

Let $D = D(V, F)$ be a homogeneous Siegel domain of type II in C^{n+m}, let $G_a(D)$ be the group of affine automorphisms of D. It is clear that $G_a(D)$ is a closed subgroup in the group of all non-singular affine transformations of the space $\mathbb{C}^{n+m}$. In proposition 4 we found the general form of the transformations in $G_a(D)$. First of