Lecture Notes of the Unione Matematica Italiana 11

Antoine Derighetti

Convolution Operators on Groups

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Lecture Notes of
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ISSN 1862-9113
ISBN 978-3-642-20655-9 e-ISBN 978-3-642-20656-6
DOI 10.1007/978-3-642-20656-6
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011932164

Mathematics Subject Classification (2000): 22DXX; 43A10; 43A20; 43A22; 43A40; 43A45; 43A15

Cover design: deblik, Berlin

Printed on acid-free paper

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Preface

Roughly speaking a convolution operator T on a group G is a linear operator on complex functions $\varphi : G \to \mathbb{C}$ that commutes with left translations

$$_g(Tf) = T(_g f).$$

Typically convolution by fixed functions gives rise to convolution operators.

To be more precise, one has to specify G and the underlying function space for T. One may suppose that G is a locally compact group with Haar measure m, and choose T to be a continuous linear endomorphism of $L^p(G) = L^p(G; m)$, where $p > 1$ is some fixed real number. It's these convolution operators that will be the subject of this book, individual cases of them as well as, for given p and G, the space $CV_p(G)$ of all of them.

The set $CV_p(G)$ is a sub Banach algebra of the Banach algebra of all continuous linear endomorphisms of $L^p(G)$. If G is abelian, it is possible to define the Fourier transform of every T in $CV_2(G)$. The Fourier transform is a Banach algebra isometry of $CV_2(G)$ onto $L^\infty(\widehat{G})$. Here, $\widehat{G}$ denotes the Pontrjagin dual of G. Moreover, $CV_p(G) \subset CV_2(G)$, this permits to define the Fourier transform of every T in $CV_p(G)$.

The case of $G = \mathbb{R}^n$ involves results of classical Fourier analysis. For instance, the fact that the Heaviside function is the Fourier transform of some $T \in CV_p(\mathbb{R})$ implies Marcel Riesz's famous theorem on the convergence in L^p of Fourier series. This convergence still holds in two variables for square summation, but not for circular summation if $p \neq 2$. This reflects the fact that the indicator function of any square is the Fourier transform of some $T \in CV_p(\mathbb{R}^2)$ but not the indicator function of the disk except if $p = 2$.

In this book, we will be mainly concerned with the investigation of $CV_p(G)$ for noncommutative groups.

If $k \in L^p(G)$ and $l \in L^{p'}(G)$, then $\overline{k} * \check{l} \in C_0(G)$ with $\|\overline{k} * \check{l}\|_\infty \leq \|k\|_p \|l\|_{p'}$.

Forming series of such functions leads to the very important Figà-Talamanca space $A_p(G)$ contained in $C_0(G)$. $A_p(G)$ is an algebra for the pointwise product.

If it is given a norm based on $\|k\|_p \|l\|_{p'}$, it becomes a Banach algebra. There is a natural duality between $CV_p(G)$ and $A_p(G)$ for a large class of locally compact groups. This duality holds for all locally compact groups if $p = 2$. It is conjectured that it holds even for all p. If G is abelian, then $A_2(G)$ turns out to be the space of Fourier transforms of $L^1(\widehat{G})$. Here, again the Fourier transform is a Banach algebra isometry of $L^1(\widehat{G})$ onto $A_2(G)$.

To every integrable function on G, and more generally to every bounded measure on G, there corresponds by convolution an operator in $CV_p(G)$. For finite groups all of $CV_p(G)$ is obtained in this manner. It is not the case for infinite groups like $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{T}$ and probably for all infinite groups. Then we may ask whether every convolution operator may be approximated by operators associated to bounded measures, and in which topology. For $p = 2$ the answer is yes under the weak operator topology. This result was obtained by Murray and von Neumann for discrete groups, by Segal for unimodular groups and finally by Dixmier for general locally compact groups. The duality between $CV_p(G)$ and $A_p(G)$ permits to answer positively for $p \neq 2$ for all amenable groups.

Let I be an ideal of the algebra $A_p(G)$. The set of points of G where all functions in I vanish will be called the cospectrum of I. An elegant formulation of the celebrated tauberian theorem of Wiener is: if G is an abelian group every ideal of $A_2(G)$ with empty cospectrum is necessarily dense in $A_2(G)$. In this book, we will show that this statement holds for every group and also every $p > 1$. The fact that the theorem of Wiener is verified on arbitrary groups is highly surprising: there are papers suggesting the impossibility of such an extension for the group of two by two invertible matrices of complex numbers!

There is a hudge amount of literature concerning the non-commutative version of the Plancherel theorem and the inversion formula for C^∞ functions with compact support on Lie groups. Such questions are, for commutative groups, very simple. An achievement of this book is the extension to non-commutative groups of theorems which are deep and difficult even for $\mathbb{Z}$, $\mathbb{T}$ or $\mathbb{R}$.

An important part of this monograph deals with the relation between $CV_p(H)$ and $CV_p(G)$, where H is a closed subgroup of G. Let i be the inclusion map of H into G. Then i induces a canonical map, also denoted i, of $CV_p(H)$ into $CV_p(G)$. For $G = \mathbb{R}$ and $H = \mathbb{Z}$, this is a famous result due to Karel de Leeuw (1965), and to Saeki (1970) for G abelian and H arbitrary closed subgroup. It is also possible to characterise the image of i in $CV_p(G)$ and to obtain in this way non-commutative analogs of a result of Reiter (1963) concerning the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$ and also to the fact that H is a set of synthesis in G (1956). The characterisation in $CV_p(G)$ of the image of $CV_p(H)$ under the map i, is a deep result due to Lohoué (1980). A large part of Chap. 7 is devoted to a detailed proof of Lohoué's result. As a consequence we obtain the extension of the Kaplansky–Helson theorem to non-abelian groups and to $p \neq 2$: for x in a arbitrary locally compact group G, every ideal of $A_p(G)$ having the cospectrum $\{x\}$ is dense in the set of all functions vanishing in x.

In the last chapter, we prove that for amenable groups $CV_p(G)$ is contained in $CV_2(G)$: this statement, compared to the commutative case, requires an entirely new approach.

The development of harmonic analysis on non-commutative groups is not just a straightforward generalization of the commutative case. It requires new ideas but it also gives rise to new problems which are far from being solved. For instance, the approximation theorem for non-amenable groups and for $p \neq 2$ is still out of reach. The investigation of the noncommutative case gives a better understanding of the commutative case! For example, instead of studying the relations between $L^\infty(\widehat{G})$ and $L^\infty(\widehat{H})$, it is more conceptual and more fruitfull to investigate the relations between the algebras $CV_2(G)$ and $CV_2(H)$.

A large part of the results presented appeared here for the first time in a book's form. The presentation is selfcontained and complete proofs are given. The prerequisities consists mostly with a familiarity with the books of Hewitt and Ross [66,67]. (Chaps. 4, 6, 8 and 10), Reiter and Stegeman [105] and Rudin [107]. Notes at the end of the volume contain additional information about results of the text.

We wish to acknowledge our indebtedness to Professor Henri Joris, who read the proofs and helped to remove some errors and obscurities. His comments have stimulated us to improve the text in several places. Those errors which do appear in the text are, of course, my own responsibility. Thanks are also due to Professor Noël Lohoué and many colleagues for encouragement and help. We would like to thank especially Professor Gerhard Racher for improvements and suggestions in relation with chapter height.

Contents

List of Symbols

We list here the symbols which are systematically used throughout the book. The numbers in parentheses refer to the paragraphs where the symbols are defined.

$\mathcal{A}_p(G)$, $A_p(G)$ (3.1)

$[f]$, $\dot{f}$ (1.1.1)

$\check{\varphi}$, ${}_a\varphi$, φ_a, $\overline{\varphi}$, $\tilde{\varphi}$, φ^*, $\tau_p\varphi$ (1.1.2)

$[f]\check{}$, $\tau_p[f]$, ${}_a[f]$, $[f]_a$ (1.1.2)

$\check{\mu}$, $\overline{\mu}$, $\tilde{\mu}$ (1.1.2)

$\mathbb{F}_2$ (1.1.4)

α_p (1.5)

β (7.1)

$\varphi * \mu$ (1.1.3)

$CV_p(G)$ (1.2)

$\widehat{T}$ (1.6)

$C(X;Y)$, $C_{00}(X;Y)$, $C(X)$, $C_{00}(X)$, $C_0(X)$ (1.1.1)

$\mathcal{L}^p(X;\mu)$, $L^p(X;\mu)$, $\mathcal{L}^p(G)$, $L^p(G)$, $<>$ (1.1.1), (1.1.2)

$\mathcal{L}^p_V(X;\mu)$, $L^p_V(X;\mu)$ (3.3)

$\mathcal{L}(L^p(X;\mu))$ (1.1.1)

$|||T|||_p$ (1.1.1)

$\lambda^p_G(\mu)$, $\lambda^p_G(f)$, $\lambda^p_G([f])$ (1.2)

$\Lambda_{\hat{G}}$ (1.3)

$\mathcal{M}^\infty(X,\mu)$, $\mathcal{M}^\infty_{00}(X,\mu)$, $\mathcal{M}^\infty(G)$, $\mathcal{M}^\infty_{00}(G)$ (1.1.1), (1.1.2)

Chapter 1
Elementary Results

We give the basic properties of the Banach algebra $CV_p(G)$. For a locally compact abelian group G we show that $CV_2(G)$ is isomorphic to $L^\infty(\widehat{G})$ and define the Fourier transform of every element of $CV_p(G)$.

1.1 Basic Notations and Basic Definitions

1.1.1 Radon Measures and Integration Theory

Let X be a topological space and Y a topological vector space. We denote by $C(X;Y)$ the vector space of all continuous maps of X into Y and by $C_{00}(X;Y)$ the subspace of all maps having compact support. We put $C(X) = C(X;\mathbb{C})$ and $C_{00}(X) = C_{00}(X;\mathbb{C})$. We denote by $C_0(X)$ the subspace of all elements of $C(X)$ vanishing at infinity.

Suppose that X is a locally compact Hausdorff space and that μ is a complex Radon measure on X. For φ an arbitrary map of X into $[0,\infty]$

$$\int_X^* \varphi(x)\,d|\mu|(x)$$

denotes the upper integral in the sense of Bourbaki ([6], p. 112, Chap. IV, Sect. 4.1, no. 3, Définition 3.) We write $\mathcal{L}^1(X,\mu)$ for the $\mathbb{C}$-vector space of all $\varphi \in \mathbb{C}^X$ which are μ-integrable. For $\varphi \in \mathcal{L}^1(X,\mu)$ the integral of φ with respect to μ is denoted $\mu(\varphi)$ or

$$\int_X \varphi(x)\,d\mu(x).$$

A. Derighetti, *Convolution Operators on Groups*, Lecture Notes of the Unione Matematica Italiana 11, DOI 10.1007/978-3-642-20656-6_1,

For $f \in \mathbb{C}^X$ or $[-\infty, \infty]^X$ locally μ-integrable we denote by $f\mu$ the Radon measure defined by

$$(f\mu)(\varphi) = \mu(f\varphi)$$

for every $\varphi \in C_{00}(X)$.

If $1 < p < \infty$ $\mathcal{L}^p(X, \mu)$ is the $\mathbb{C}$-vector space of all $\varphi \in \mathbb{C}^G$ such that φ is μ-measurable and $|\varphi|^p$ is μ-integrable. If $f \in \mathbb{C}^X$ we denote by $[f]$ the set of all $g \in \mathbb{C}^X$ with $g(x) = f(x)$ μ-almost everywhere and by $\dot{f}$ the set of all $g \in \mathbb{C}^X$ with $g(x) = f(x)$ locally μ-almost everywhere.

Suppose $1 \leq p < \infty$. For $f \in \mathbb{C}^X$ or for an arbitrary map of X into $[-\infty, \infty]$ we put

$$N_p(f) = \left(\int_X^* |\varphi(x)|^p d|\mu|(x) \right)^{1/p} .$$

N_p is a semi-norm on $\mathcal{L}^p(X, \mu)$. With respect to this semi-norm $\mathcal{L}^p(X, \mu)$ is complete. For $f \in \mathcal{L}^p(X, \mu)$ we set

$$\|[f]\|_p = N_p(f) \quad \text{and} \quad L^p(X, \mu) = \left\{ [f] \middle| f \in \mathcal{L}^p(X, \mu) \right\}$$

which is a Banach space for the norm $\| \|_p$.

For an arbitrary map f of X into $[-\infty, \infty]$ we put

$$M_\infty(f) = \inf \left\{ \alpha \in [-\infty, \infty] \middle| f(x) \leq \alpha \quad \text{locally } \mu\text{-almost everywhere} \right\}.$$

For $f \in \mathbb{C}^X$ we set $N_\infty(f) = M_\infty(|f|)$. For f a bounded complex function we set

$$\|f\|_u = \sup \left\{ |f(x)| \middle| x \in X \right\}.$$

Let $\mathcal{M}^\infty(X, \mu)$ be the $\mathbb{C}$-subspace of $\mathbb{C}^X$ of all functions which are μ-measurable and bounded and $\mathcal{L}^\infty(X, \mu)$ the $\mathbb{C}$-subspace of $\mathbb{C}^X$ of all functions which are locally μ-almost everywhere equal to a function of $\mathcal{M}^\infty(X, \mu)$. Then N_∞ is a semi-norm on $\mathcal{L}^\infty(X, \mu)$, with respect to this semi-norm $\mathcal{L}^\infty(X, \mu)$ is complete. By definition

$$L^\infty(X, \mu) = \left\{ \dot{f} \middle| f \in \mathcal{M}^\infty(X, \mu) \right\}.$$

With the norm

$$\|\dot{f}\|_\infty = N_\infty(f),$$

$L^\infty(X, \mu)$ is a Banach space. We denote by $\mathcal{M}^\infty_{00}(X, \mu)$ the subspace of all $f \in \mathcal{M}^\infty(X, \mu)$ with compact support.

Finally let $M^1(X)$ be the space of all complex bounded Radon measures on X. For $\mu \in M^1(X)$ we put

$$\|\mu\| = \sup\left\{|\mu(\varphi)| \,\middle|\, \varphi \in C_{00}(X), \|\varphi\|_u \leq 1\right\}.$$

Then $\|\,\|$ is a norm on $M^1(X)$.

Let $1 \leq p < \infty$ we put $p' = p/(p-1)$ if $p > 1$ and $p' = \infty$ if $p = 1$. For $f \in \mathcal{L}^p(X,\mu)$ and $g \in \mathcal{L}^{p'}(X,\mu)$ we set

$$\langle [f], [g] \rangle = \int_X f(x)\overline{g(x)}d\mu(x)$$

if $p > 1$, and if $p = 1$

$$\langle [f], \dot{g} \rangle = \int_X f(x)\overline{g(x)}d\mu(x).$$

The function $\langle , \rangle$ is a sesquilinear form on $L^p(X,\mu) \times L^{p'}(X,\mu)$.

Let $\mathcal{L}(L^p(X,\mu))$ be the linear space of all continuous endomorphisms of $L^p(X,\mu)$. For $T \in \mathcal{L}(L^p(X,\mu))$, $|||T|||_p$ is the bound of the operator T:

$$|||T|||_p = \sup\left\{\|Tf\|_p \,\middle|\, f \in L^p(X,\mu), \|f\|_p \leq 1\right\}.$$

For the composition of the operators, $\mathcal{L}(L^p(X,\mu))$ is a Banach algebra.

For V a topological vector space, V' denotes the dual of V. If $(V, \|\,\|_V)$ is a normed vector space, and if $F \in V'$ we put

$$\|F\|_{V'} = \sup\left\{|F(v)| \,\middle|\, v \in V, \|v\|_V \leq 1\right\}.$$

This norm makes V' into a Banach space.

1.1.2 Locally Compact Groups

Let G be a group. For a non-empty set Y, φ a map of G into Y, a and $x \in G$ we put

$$\check{\varphi}(x) = \varphi(x^{-1}),\ {}_a\varphi(x) = \varphi(ax) \quad \text{and} \quad \varphi_a(x) = \varphi(xa).$$

Let now be G a locally compact group. We always suppose that the topology of G is Hausdorff. We recall that there is a nonzero positive Radon measure m_G on G such that

$$m_G(\varphi) = m_G({}_a\varphi) = m_G(\varphi_a)\Delta_G(a) = m_G\left(\check{\varphi}\check{\Delta}_G\right)$$

for every $\varphi \in C_{00}(G)$ and every $a \in G$. Here Δ_G is a continuous homomorphism of G into the multiplicative group $(0,\infty)$. Up to a multiplicative real number, the measure m_G is unique. The measure m_G is called a left invariant Haar measure of G.

The function Δ_G does not depend of the choice of the measure m_G. This function is called the modular function of G. If G is compact we suppose that $m_G(1_G) = 1$.

Let us present some basic examples.

(a) If $G = \mathbb{T}$

$$m_{\mathbb{T}}(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta}) d\theta$$

for $\varphi \in C(\mathbb{T})$.

(b) For $G = \mathbb{R}$ we may choose

$$m_{\mathbb{R}}(\varphi) = \int_{-\infty}^{\infty} \varphi(x) dx$$

for every $\varphi \in C_{00}(\mathbb{R})$.

(c) Take now an arbitrary group G. Consider on G the discrete topology. This locally compact group is denoted G_d. Suppose at first that G is finite. Then $C_{00}(G_d) = \mathbb{C}^G$. We have

$$m_{G_d}(\varphi) = \frac{1}{|G|} \sum_{x \in G} \varphi(x)$$

for every $\varphi \in C(G_d)$. If G is infinite then $C_{00}(G_d)$ is the subspace of $\mathbb{C}^G$ of all functions having a finite support and we may choose

$$m_{G_d}(\varphi) = \sum_{x \in G} \varphi(x)$$

for every $\varphi \in C_{00}(G_d)$.

In all these examples $\Delta_G = 1$.

(d) Let G be the group of matrices

$$\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$$

where $x, y \in \mathbb{R}, x \neq 0$, with the topology induced by $\mathbb{R}^2$. Then we may choose

$$m_G(\varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(x, y)}{x^2} dx dy.$$

One has

$$\Delta_G\left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\right) = \frac{1}{x^2}.$$

In the examples (a), (b) and (d), the integral on the right hand side is the Riemann integral.

Let again G be any locally compact group. For $\varphi \in \mathbb{C}^G$ we set

$$\overline{\varphi}(x) = \overline{\varphi(x)}, \quad \tilde{\varphi}(x) = \overline{\varphi(x^{-1})} \quad \text{and} \quad \varphi^*(x) = \overline{\varphi(x^{-1})}\Delta_G(x^{-1}).$$

Let be μ a complex Radon measure on G. Then we define the three Radon measures $\check{\mu}$, $\overline{\mu}$ and $\tilde{\mu}$ by

$$\check{\mu}(\varphi) = \mu(\check{\varphi}), \quad \overline{\mu}(\varphi) = \overline{\mu(\overline{\varphi})} \quad \text{and} \quad \tilde{\mu}(\varphi) = \overline{\mu(\tilde{\varphi})}$$

where $\varphi \in C_{00}(G)$. For $f \in \mathbb{C}^G$ and $1 \leq p < \infty$ we also put

$$\tau_p(f)(x) = f(x^{-1})\Delta_G(x^{-1})^{1/p}.$$

For f m_G-integrable we set

$$m_G(f) = m(f) = \int\limits_G f(x)dx, \mathcal{L}^p(G) = \mathcal{L}^p(G, m_G), L^p(G) = L^p(G, m_G)$$
$$(1 \leq p \leq \infty)$$

and

$$\mathcal{M}_{00}^{\infty}(G) = \mathcal{M}_{00}^{\infty}(G, m_G).$$

For $f \in \mathbb{C}^G$ we put:

$$[f]\check{} = [\check{f}\,] \quad \text{and for} \quad 1 \leq p < \infty \quad \tau_p[f] = [\tau_p f],$$

for $a \in G$ we also put

$$_a[f] = [_a f] \quad \text{and} \quad [f]_a = [f_a].$$

Clearly τ_p is an isometric involution of the Banach space $L^p(G)$ for $1 \leq p < \infty$.

1.1.3 Convolution of Measures and Functions

Formally the convolution $\mu * \nu$ of the two Radon measures μ and ν on the locally compact group G is defined by

$$(\mu * \nu)(f) = \int\limits_{G \times G} f(xy)d\mu(x)\,d\nu(y)$$

whenever the double integral converges absolutely for all $f \in C_{00}(G)$. This is the case for example if one of the two given measures has compact support, or if both of them are bounded. For $\mu = gm$ we have

$$(gm * \nu)(f) = \int_{G\times G} f(xy)g(x)dxd\nu(y) = \int_G \left(\int_G f(xy)g(x)dx\right) d\nu(y)$$
$$= \int_G \Delta_G(y^{-1}) \left(\int_G f(x)g(xy^{-1})dx\right) d\nu(y) = ((g * \nu)m)(f)$$

where we define

$$(g * \nu)(x) = \int_G g(xy^{-1})\Delta_G(y^{-1})d\nu(y).$$

Similarly we get $\mu * hm = (\mu * h)m$ if we define

$$(\mu * h)(y) = \int_G h(x^{-1}y)d\mu(x).$$

Putting here $\mu = gm$ we set $g * h = gm * h$ and thus

$$(g * h)(y) = \int_G g(yx)h(x^{-1})dx = \int_G g(yx^{-1})h(x)\Delta_G(x^{-1})dx.$$

We refer to [10] Chapter 8 for a detailed exposition of these questions.

1.1.4 Amenable Groups

A locally compact group G is said to be amenable if there is a linear functional $\mathcal{M}$ on the vector space $C^b(G)$ of all continuous bounded complex valued functions on G such that $\mathcal{M}(\varphi) \geqslant 0$ if $\varphi \geqslant 0$, $\mathcal{M}(1_G) = 1$ and $\mathcal{M}({}_a\varphi) = \mathcal{M}(\varphi)$ for every $a \in G$.

We only recall that compact, abelian or solvable groups are amenable. But $SL_2(\mathbb{R})$, the group of two by two real matrices with determinant one, and the free group $\mathbb{F}_2$ of two generators are not amenable. Every closed subgroup of an amenable group is amenable. If G is a locally compact group and H a closed normal subgroup, and if H and G/H are amenable, then so is G.

They are many properties equivalent to the amenability. We will use the following one: for every $\varepsilon > 0$ and for every compact subset K of G there is $s \in C_{00}(G)$ with $s \geqslant 0$, $N_1(s) = 1$ and $N_1({}_ks - s) < \varepsilon$ for every $k \in K$. We refer to Chap. 8 of [105] for detailed proofs of all these assertions.