$$\sqrt{2}$$

The Square Root of 2

A Dialogue Concerning a Number and a Sequence

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Why, sir, if you are to have but one book with you on a journey, let it be a book of science. When you have read through a book of entertainment, you know it, and it can do no more for you; but a book of science is inexhaustible. . . .

—James Boswell
Journal of a Tour to the Hebrides with Samuel Johnson

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Prologue

You may think of the dialogue you are about to read, as I often did while writing it, as being between a "master" and a "pupil"—the master in his middle years, well-versed in mathematics and as devoted and passionate about his craft as any artist is about his art; the pupil on the threshold of adulthood, articulate in speech, adventuresome of mind, and enthusiastically receptive to any knowledge the more learned teacher may care to impart.

Their conversation—the exact circumstances of which are never described—is initiated by the master, one of whose tasks is to persuade his disciple that the concept of number is more subtle than might first be imagined. Their mathematical journey starts with the teacher guiding the student, by way of questions and answers, through a beautifully simple geometrical demonstration (believed to have originated in ancient India), which establishes the existence of a certain number, the understanding of whose nature is destined to form a major part of the subsequent discussion between the enquiring duo.

Strong as the master's motivation is to have the younger person glimpse a little of the wonder of mathematics, stronger still is his desire to see that his protégé gradually becomes more and more adept at mathematical reasoning so that he may experience the pure pleasure to be had from simply "finding things out" for himself. This joy of discovery is soon felt by the young learner, who having embarked upon an exploration, is richly rewarded when, after some effort, he chances upon a sequence of numbers that he surmises is inextricably linked to the mysterious number lately revealed by the master. Enthralled by this fortunate occurrence, he immediately finds himself in the grip of a burning curiosity to know more about this number and its connection with the sequence that has already captivated him. Thus begins this tale told over five chapters.

I have made every effort to have the first four chapters as self-contained as possible. The use of mathematical notation is avoided whenever words can achieve the same purpose, albeit in a more lengthy manner. When mathematical notation is used, nothing beyond high school algebra of the simplest kind is called on, but in ways that show clearly the need for this branch of mathematics. While the algebra used is simple, it is often clever, revealing that a few tools handled with skill can achieve a great deal. If readers were to appreciate nothing more than this aspect of algebra—its power to prove things in general—then this work will not have been in vain.

Unfortunately, to have the fifth chapter completely self-contained would have meant sacrificing exciting material, something I didn't wish to do, preferring to reward the reader for the effort taken to reach this point, when it is hoped he will understand enough to appreciate the substance of what is being related.

Throughout the dialogue, so as to distinguish between the two speakers, the following typographical conventions are used:

The Master's Voice—assured, but gently persuasive—is set in this mildly bold typeface, and is firmly fixed at the left edge of the column.

> The Pupil's Voice—deferential, but eager and inquiring—is set in this lighter font, and is moved slightly inward from the margin.

The best conversations between teachers and students are both serious and playful, and my hope is that the readers of this book will sense that something of that spirit, of real learning coupled with real pleasure, coexist in this dialogue.

David Flannery
September, 2005

$$\sqrt{2}$$

CHAPTER 1

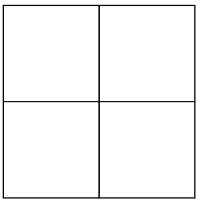
Asking the Right Questions

I'd like you to draw a square made from four unit squares.

A unit square is one where each of the sides is one unit long?

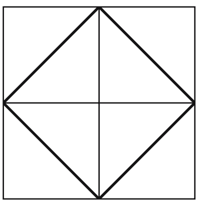
Yes.

Well, that shouldn't be too hard.



Will this do?

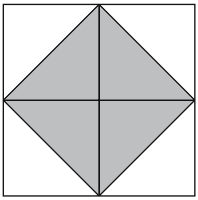
Perfect. Now let me add the following diagonals to your drawing.



You see that by doing this a new square is formed.

I do. One that uses a diagonal of each of the unit squares for its four sides.

Let's shade this square and call it the "internal" square.



Now, I want you to tell me the area of this internal square.

Let me think. The internal square contains exactly half of each unit square and so must have half the area of the large square. So it has an area of 2 square units.

Exactly. Now, what is the length of any one of those diagonals that forms a side of the internal square?

Off-hand I don't think I can say. I know that to get the area of a rectangular region you multiply its length by its breadth.

"Length by breadth," as you say, meaning multiply the length of one side by the length of a side at right angles to it.

So, for a square, this means that you multiply the length of one side by itself, since length and breadth are equal.

Yes.

But where does this get me? As I said, I don't know the length of the side.

As you say. But if we let *s* stand for the length of one of the sides, then what could you say about *s*?

I suppose there is no way that we could have this little chat without bringing letters into it?

There is, but at the cost of the discussion being more longwinded than it need be. Incidentally, why did I chose the letter *s*?

Because it is the initial of the word side?

Precisely. It is very common to use the initial of the word describing the quantity you're looking for.

So *s* stands for the length of the side of the internal square. I hope you are not going make me do algebra.

Just a very small amount—for the moment. So can you tell me something about the number *s*?

When you multiply *s* by itself you get 2.

Exactly, because the area of the internal square is 2 (squared units). Do you recall that $s \times s$ is often written as s^2?

I do. My algebra isn't *that* rusty.

So you are saying that the number s "satisfies" the equation:

$$s^2 = 2$$

In words, "s squared equals two."

Okay, so the number s when multiplied by itself gives 2. Doesn't this mean that s is called the square root of 2?

Well, it would be more accurate to say that s is *a* square root of 2. A number is said to be a square root of another if, when multiplied by itself, it gives the other number.

So 3 is a square root of 9 because $3 \times 3 = 9$.

As is -3, because $-3 \times -3 = 9$ also.

But most people would say that the square root of 9 is 3.

True. It is customary to call the positive square root of a number its square root. And since s is the length of the side of a square, it is obviously a positive quantity, so we may say . . .

. . . that s is the square root of 2.

Sometimes, we simply say "root two," it being understood that it's a square root that is involved.

And not some other root like a cube root?

Yes. Now the fact that 3 is the square root of 9 is often expressed mathematically by writing $\sqrt{9} = 3$.

I've always liked this symbol for the square root.

It was first used by a certain Christoff Rudolff in 1525, in the book *Die Coss*, but I won't go into the reasons why he chose it.

Can we say goodbye to s and write $\sqrt{2}$ in its place from now on? [See chapter note 1.]

If we want to, but we'll still use s if it serves our purposes.

So we have shown that the diagonal of a unit square is $\sqrt{2}$ in length.

Indeed we have. This wonderful way of establishing the existence of the square root of 2 originated in India thousands of years ago. [See chapter note 2.]

You'd have to say that it is quite simple.

Which makes it all the more impressive.

So what number is $\sqrt{2}$?

As the equation $s^2 = 2$ says, it is the number that, when multiplied by itself, gives 2 exactly. This means no more or no less than what the equation

$$\sqrt{2} \times \sqrt{2} = 2$$

says it means: $\sqrt{2}$ is the number that when multiplied by itself gives 2.

I know, but what number does $\sqrt{2}$ actually stand for? I mean $\sqrt{16} = 4$, and 4 is what I would call a tangible number.

I understand. You have given me a concrete value for $\sqrt{16}$, namely the number 4. You want me to do the same for $\sqrt{2}$, that is, to show you some number of a type with which you are familiar, and that when squared, gives 2.

Exactly. I'm simply asking what the concrete value of s is, that makes $s^2 = 2$.

I can convince you quite easily that $\sqrt{2}$ is not a natural number.

The natural numbers are the ordinary counting numbers, 1,2,3, and so on.

Precisely.

Even though 2 itself is a natural number? The natural numbers 9 and 16 have square roots that are also natural numbers.

That's true, they do.

But you are saying that 2 doesn't.

I am. One way of seeing this is to write the first few natural numbers in order of increasing magnitude in a line, and beneath them on a second line write their corresponding squares:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \ldots \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 \ldots \end{array}$$

The three dots, or ellipsis, at the end of a line means that the pattern continues without stopping.

Well, I can see straight away that the number 2 is missing from the second row.

As are

$$3, \quad 5, \quad 6, \quad 7, \quad 8, \quad 10, \quad 11, \quad 12, \quad 13, \quad 14, \quad 15, \quad 17, \ldots$$

I would say that there are a lot more numbers missing than are present.

Yes, in a sense "most" of the natural numbers are absent from this second line. The numbers 1, 4, 9, 16, . . . that appear on it are known as the *perfect squares*.

And those numbers that are missing from this line are not perfect squares?

Correct: 49 is a perfect square but 48 is not.

I think I see now why there is no natural number squaring to 2. The first natural number squares to 1 while the second natural number squared is 4, so 2 gets skipped over.

That's about it.

All right. It is fairly obvious, now at any rate, that there is no natural number that squares to 2, but surely there is some fraction whose square is 2?

By fraction, you mean a common fraction where one whole number is divided by another whole number?

That's what I mean, $\frac{7}{5}$, for example. Are there other types of fractions?

There are, but when we say "fraction" we mean one whole number divided by another one. The number being divided is the numerator and the one doing the dividing is called the denominator.

The number on the top is the numerator and the number on the bottom is the denominator.

That's it exactly. In your example, the whole number 7 is the numerator while the whole number 5 is the denominator.

Now mustn't there be some fraction close to this one that squares to give 2 exactly?

Why did you say *close* to this one?

Because my calculator tells me that $\frac{7}{5}$ is 1.4 in decimal form; and when I multiply this by itself I get 1.96, which is fairly close to 2.

Agreed. Let me show you how we can see this for ourselves without a calculator but using a little ingenuity instead.

Since

$$\begin{aligned}\left(\frac{7}{5}\right)^2 &= \frac{49}{25} \\ &\overset{!}{=} \frac{50-1}{25} \\ &= \frac{50}{25} - \frac{1}{25} \\ &= 2 - \frac{1}{25}\end{aligned}$$

we can say that the fraction $\frac{7}{5}$ when squared underestimates 2 by the amount $\frac{1}{25}$.

And according to my calculator $\frac{1}{25} = 0.04$, which is just $2 - 1.96$. By the way, why did you put the exclamation point over the second equals sign?

To indicate that the step being taken is quite a clever one.

It certainly wouldn't have occurred to me, which I know is not saying much.

Well, I don't lay any claim to originality for taking this step. I have seen many similar such tricks used by others in the past and, after all, I knew what it was I wanted to show.

At least I can see why it's clever.

Good. Why?

By writing the numerator 49 as $50 - 1$, you were able to divide the 50 by 25 to get 2 exactly and the 1 by 25 to get $\frac{1}{25}$ as the measure of the underestimate.

Desert Island Math

A useful trick if you're stranded on a desert island without any calculating devices other than your own poor head.

Pure do-it-yourself mathematics! I suppose using a calculator to get the value of something you wouldn't be able to calculate for yourself is a form of cheating?

Do you mean like asking for the decimal expansion of $\sqrt{2}$, for example?

Well, something like that. I wouldn't have a clue how to get the decimal expansion of $\sqrt{2}$ using my own very limited powers.

$$\begin{array}{r} 1.4 \\ 5\,\overline{)\,7.00} \\ \underline{5} \\ 20 \\ \underline{20} \\ 00 \end{array}$$

I'm sure you do your mental abilities an injustice. If we know and understand how to get a decimal expansion of a number "by hand," then we don't contravene the DIY philosophy if we use a calculator to save labor.

Are you saying that because I know how to get the decimal expansion of $\frac{7}{5}$ or $\frac{3}{11}$ by long division, even though I wouldn't like to be pressed on why the procedure works, I may use a calculator to avoid the "donkey work" involved with such a task?

algorithm: step-by-step procedure

I think we'll let this be a policy. We'll assume that if we were put to it we could explain to ourselves and others the "ins and outs" of the long-division algorithm.

Of course, completely!

$$\begin{array}{r} 0.272\ldots \\ 11\,\overline{)\,3.000} \\ \underline{22} \\ 80 \\ \underline{77} \\ 30 \\ \underline{22} \\ 80\ldots \\ \vdots \end{array}$$

Decimal expansions, or "decimals" as we often say for short, have certain advantages, one being that they convey the magnitude of a number more readily than their equivalent fractions do. When a number is expressed in decimal form, it is easy to say geometrically where it is located on the number line. No matter how long the decimal expansion of a number may be, we still know between which two whole numbers it lies on this number line:

So we can see quite easily from 1.4 that it is a number between 1 and 2, whereas it is not as easy to see this from the fraction $\frac{7}{5}$.

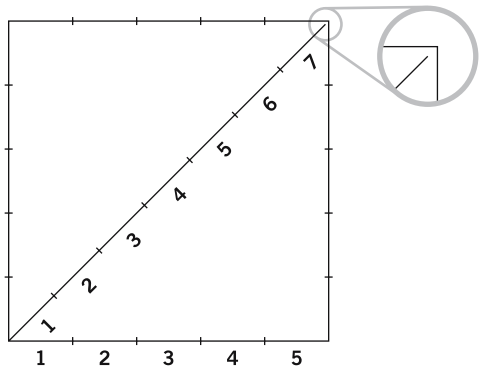
The fraction $\frac{7}{5}$ is perhaps too simple. It is not too difficult to mentally determine the two whole numbers between which it is located on the number line, but who can say without resorting to a calculation where the fraction $\frac{103993}{33102}$ is positioned on the same line?

I see the point, or should I say I do not see the (decimal) point!

Hmm! Speaking of the fraction $\frac{7}{5}$, you might like to get a box of matches and construct a square with five matches on each side.

Does this mean that the five matches between them make up the unit-length?

You can certainly think of it this way, if you like. Now you'll find that seven matches will fit along the diagonal:



These seven matches do not stretch the full length of the diagonal since $\frac{7}{5}$ underestimates $\sqrt{2}$.

That they don't is barely visible.

True, but the gap is there.

This is a rather neat way of visualising $\frac{7}{5}$ as an approximation to $\sqrt{2}$.

Yes it is, isn't it? Looked at another way it says that the ratio $7:5$ is close to the ratio $\sqrt{2}:1$. Now, where were we?

Looking for a fraction that squares to 2.

Indeed, so let's continue the quest. Any further thoughts?

There must be some fraction a little bit bigger than $\frac{7}{5}$ that squares to give 2 exactly.

Well, there are lots of fractions just a little bit bigger than $\frac{7}{5}$.

I know. Isn't there an infinity of fractions between 1.4 and 1.5 alone?

Yes, but that this is so we can leave for another time. Why do you mention 1.5?

Simply because $(1.4)^2 = 1.96$ is less than 2 while $(1.5)^2 = 2.25$ is greater than 2.

So?

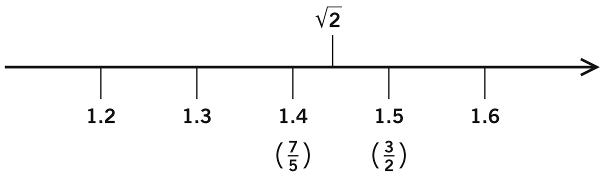
Doesn't this mean that the square root of 2 lies between these two values?

The symbol $<$ means "less than."

It does. In fact since $1.5 = \frac{3}{2}$ we may write that

$$\frac{7}{5} < \sqrt{2} < \frac{3}{2}$$

Let me display this arithmetic "inequality" on the number line:



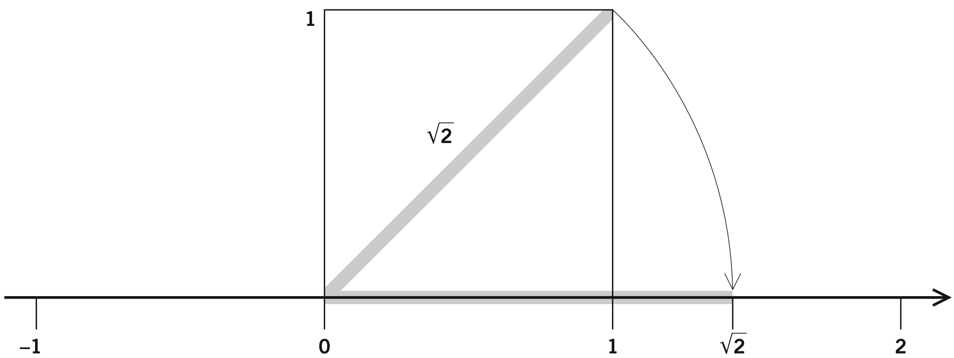
Notice that I have placed $\sqrt{2}$ to the right of 1.4 and closer to 1.4 than to 1.5 because $\frac{3}{2}$ squared overestimates 2 by $\frac{1}{4}$, which is much more than the $\frac{1}{25}$ by which $\frac{7}{5}$ squared underestimates $\sqrt{2}$.

But how do you locate $\sqrt{2}$ on the number line if you don't know what fraction it is?

A good question. The answer is that you do so geometrically.

I'd like to see how.

It's easy to construct a unit square geometrically on the interval that stretches between 0 and 1:



Now imagine the diagonal with one end at 0 and of length $\sqrt{2}$ being rotated clockwise about the point 0 until its other end lies on the number line.

At a point $\sqrt{2}$ from 0. Very smart.

Of course, this is an ideal construction where everything can be done to perfection.

I understand. It is the method that counts.

Yes.

An Exploration

But to return to the point I was making: surely among the infinity of fractions lying between 1.4 and 1.5 there is one that squares to give 2 exactly.

Well if there is, how do you propose finding it?

That's what is bothering me.

I'm sure you'll agree that it's not wise to begin checking fraction after fraction in this infinity of fractions without having some kind of plan.

Absolutely, it could take forever. What would you suggest?

Thinking about the problem a little to see if we can find some systematic way of attacking it.

Sounds as if we are about to go into battle.

A mental battle. Let us begin our campaign by examining the implications of expressing the number $\sqrt{2}$ as a fraction.

This could get interesting. What are you going to call this fraction?

Well, since we don't know it, at least not yet, we must keep our options open. One way of doing this is to use distinct letters, one to stand for its numerator and the other for its denominator.

Here comes some more algebra.

Only a little, used as scaffolding as it were, just to get us started.

Well, I'll stop you if I think I'm losing the drift of the discussion.

Let's call the numerator of the fraction *m* and the denominator *n*.

So if the fraction were $\frac{7}{5}$, which I know it is not, then m would be 7 and n would equal 5.

Or put slightly differently, if $m = 7$ and $n = 5$ then

$$\frac{m}{n} = \frac{7}{5}$$

I'm with you.

Now if

$$\sqrt{2} = \frac{m}{n}$$

then

$$\sqrt{2} \times \sqrt{2} = \frac{m}{n} \times \frac{m}{n}$$

Agreed?

I think so. You are simply squaring both sides of the original equation.

I am, and I do so in this elaborate manner to highlight the presence of $\sqrt{2} \times \sqrt{2}$.

Which by definition is 2.

Yes, a simple but vital use of the defining property of $\sqrt{2}$, which allows us to write that

$$2 = \left(\frac{m}{n}\right)^2$$

We can turn this equation around and write

$$\left(\frac{m}{n}\right)^2 = 2$$

to put the emphasis on the fraction $\frac{m}{n}$. What is the equation saying about $\frac{m}{n}$?

That its square is 2.

Exactly. And since

$$\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

we can say that

$$\frac{m^2}{n^2} = 2$$

or that

$$m^2 = 2n^2$$

So this equation is a consequence of writing $\sqrt{2}$ as $\frac{m}{n}$?

It is indeed. Now let us see what we can learn from it.

I'll leave this to you.

I'm sure it won't be long before you join in. For one thing, $m^2 = 2n^2$ tells us is that if we are to find a fraction that is equal to $\sqrt{2}$, then we must find two perfect squares, one of which is twice the other.

What are perfect squares again? Oh, I remember, 1, 4, 9, 16, . . .

That's right, a perfect number is one that is the square of a natural number.

Well, this is a task that I can definitely undertake.

Be my guest.

Why don't I make out a list of the first twenty squares along with their doubles and see if I can find a match between some square and the double of some other square.

An excellent plan. Nothing like a bit of "number crunching," as it's called, to really get one thinking.

Of course, I'm going to use a calculator just to speed things up.

Naturally. Nobody doubts that you can multiply one number by itself.

Here's the table I get:

Natural Number	Number Squared	Twice Number Squared
1	1	2
2	4	8
3	9	18
4	16	32
5	25	50
6	36	72
7	49	98
8	64	128
9	81	162
10	100	200
11	121	242
12	144	288
13	169	338
14	196	392
15	225	450
16	256	512
17	289	578
18	324	648
19	361	722
20	400	800

The three columns show, in turn, the first twenty natural numbers, their squares, and twice these squares.

Great. We can think of the second column as corresponding to m^2 numbers and the third column as corresponding to numbers of the form $2n^2$.

I'm not sure I understand what you are saying here.

I'll explain by example. We may think of the number 196 in the second column as being an m^2 number, where $m = 14$, while we may consider the number 450 in the third column as being a $2n^2$ number, where $n = 15$.

Let me test myself to see if I have got the idea. I can think of 16 in the second column as an m^2 number with $m = 4$, while I can think of the 648 in the third column as corresponding to $2n^2$, with $n = 18$, because $2(18)^2 = 648$. Do I pass?

With honors. Now if you can find an entry in the second column that matches an entry in the third column, you will have found values for m and an n which make $m^2 = 2n^2$ and so you'll have a fraction $\frac{m}{n}$ equal to $\sqrt{2}$.

As easy as that? So fingers crossed as I look at each entry of the second column of this table and then look upwards from its location along the third column for a possible match.

Of course! A time-saving observation. As you say, you need only look upwards because the corresponding entries in the third column are bigger than those in the second.

Unfortunately, I can't find a single entry in the second column that is equal to any entry in the third column.

So the second and third columns have no element in common.

Not that I can see. I'm going to experiment a little more by calculating the next ten perfect squares along with their doubles.

Good for you.

This time I get:

Natural Number	Number Squared	Twice Number Squared
21	441	882
22	484	968
23	529	1058
24	576	1152
25	625	1250
26	676	1352
27	729	1458

Natural Number	Number Squared	Twice Number Squared
28	784	1568
29	841	1682
30	900	1800

I realize that this is not much of an extension to the previous table.

Maybe, but perhaps you'll get a match this time.

I'm scanning the second column to see if any entry matches anything in the previous third column or the new third column.

Any luck?

I'm afraid not. However, I notice that there are some near misses in the first table.

What do you mean by "near misses"?

A discovery?

There are entries in the second column that are either just 1 less or 1 more than an entry in the third column.

I'm more than curious; please elaborate.

Well, take the number 9 in the second column. It is 1 more than the 8 in the third column.

True. Any others?

There's a 49 in the second column that is 1 less than the 50 appearing in the third column.

Again, true. Any more?

Yes. There's a 289 in the second column and a 288 in the third column.

Again, as you observed, with a difference of 1 between them. Did you find any more examples?

Not that I can see in these two tables, except, of course, at the very beginning. There's a 1 in the second column and a 2 in the third column.

Indeed there is.

But I don't know what to make of these near misses.

However, you seem to have hit upon something interesting, exciting even, so let's take a little time out to mull over your observations.

Fine by me, but you'll have to do the thinking.

Why don't we look at the case of the 9 in the second column and the 8 in the third column. What is the m number corresponding to this 9 in the second column, and what is the n number corresponding to the 8 in the third column?

Let me think. I would say that $m = 3$ and that $n = 2$.

And you'd be right. Your observation tells us that

$$m^2 = 2n^2 + 1$$

where $m = 3$ and $n = 2$.

Because $3^2 = 2(2)^2 + 1$?

Exactly. Now let us move on to the case of the number 49 in the second column and the 50 in the third column.

Here the $m = 7$ and $n = 5$ since $2(5)^2 = 2(25) = 50$.

This time

$$m^2 = 2n^2 - 1$$

where $m = 7$ and $n = 5$.

Can I try the next case?

By all means.

The number 289 corresponds to $m = 17$ since in this case $m^2 = 289$. On the other hand, the number 288 corresponds to $n = 12$ since $2(12)^2 = 2(144) = 288$.

No argument there.

This time

$$m^2 = 2n^2 + 1$$

where $m = 17$ and $n = 12$.

So we're back to 1 over.

There seems to be an alternating pattern with these pairs of near misses.

There does indeed. For the sake of completeness, you should look at the first case.

You mean the case with 1 in the second column and 2 in the third column?

None other; the smallest case, so to speak.

Okay. Here $m = 1$ and $n = 1$.

And what is the value of $m^2 - 2n^2$ on this occasion?

This time

$$m^2 = 2n^2 - 1$$

Does this fit the alternating pattern?

It does.

Which is great.

But returning to the original reason for constructing the tables, I haven't found a single square among the first thirty perfect squares that is equal to twice another square.

True, and that means that, as of yet, you haven't found a fraction $\frac{m}{n}$ that squares to 2. But, on the other hand, you have found a number of very interesting fractions.

I have? I would have thought that I've only found pairs of natural numbers that are within 1 of each other.

In a sense, you could say that. But you actually have discovered fractions with the property that the square of their numerator is within 1 of double the square of their denominator.

I'm afraid you'll have to elaborate.

Of course. You remember we said, when you observed that 9 in the second column was 1 greater than the 8 in the third column, that the 9 corresponded to m^2 where $m = 3$, while the 8 corresponded to $2n^2$ where $n = 2$?

I do.

Furthermore, $m^2 - 2n^2 = 1$, in this case.

That's correct.

Suppose now that we form the fraction

$$\frac{m}{n} = \frac{3}{2}$$

Then can't we say that the equation $m^2 - 2n^2 = 1$ tells us that this fraction is such that the square of its numerator is 1 more than twice the square of its denominator?

It seems to. I'll have to think a little more about this. Yes: $3^2 = 9$ and $2(2)^2 = 8$.

Try another one. Ask yourself, "What fraction is associated with the observation that the 49 in the second column is 1 less than the 50 in the third column?"

Okay. Here $m = 7$ and $n = 5$, so the fraction is $\frac{7}{5}$, right?

Absolutely. Now what can you say about the numerator and denominator of this fraction?

That the square of the numerator is 1 less than twice the square of its denominator.

Exactly.

I think I understand now. You are saying that every time we observe the near miss phenomenon we actually find a special fraction.

Yes. You looked for a fraction whose numerator squared would match twice its denominator squared; you didn't find one, but instead you found fractions whose numerators squared are within 1 of twice their denominators squared.

That's a nice way of looking at it.

Often when you look for something specific you chance upon something else.

So I suppose you could say that I found the next best thing.

I think we can say this, and not a bad reward for your labors.

Actually, I'm really curious to know if there are any more than just these four misses and to see if the plus or minus pattern continues to hold.

Let's hope so. Why don't we do a little more exploring?

I'd be happy to but shouldn't we stick to our original mission of finding a difference of exactly 0?

Very nicely put. Finding an m and n such that $m^2 = 2n^2$ means that the difference $m^2 - 2n^2$ would be 0.

Thanks.

However, I think we'll indulge ourselves and investigate your observation about near misses a little further, particularly as it looks so promising.

Okay. I'll extend my tables and then go searching.

You could do that, but it might be an idea to look more carefully at what you have already found.

Like good scientists would.

As you say. Begin by cataloguing the specimens found to date and examine them carefully for any clues.

Will do.

Time to Reflect

Beginning with the smallest, and listing them in increasing order, the fractions are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$$

Not many as of yet, but tantalizing.

What secrets do they hold, if any?

Indeed. Can you spot some connection between them?

Just like one of those sequence puzzles, "What is the next number in the sequence?" except here it looks harder because these are fractions and not ordinary numbers.

A puzzle certainly, but one we have encountered quite naturally.

And not just made up for the sake of it.

Yes, something like that.

I hope this is an easy puzzle.

It is always best to be optimistic so I advise that you say to yourself, "This is sure to be easy," and look for simple connections.

Optimism it is then, but where to start?

It is often a good idea to begin by examining a pair of terms some way out along a sequence rather than at the very beginning of it.

Right. On that advice I'll see if I can spot a connection between

$$\frac{7}{5} \quad \text{and} \quad \frac{17}{12}$$

and if I think I have found one, I'll check it on the earlier fractions.

Very sensible. Happy hunting!

I think I'll begin by focusing on the denominator 12 of the fraction $\frac{17}{12}$.

Following a very definite line of inquiry, as they say.

I think I have spotted something already.

Which is?

That $12 = 7 + 5$, the next denominator looks as if it might be the sum of the numerator and denominator of the previous fraction.

If it's true, it will be a big breakthrough. I must say that was pretty quick.

I must check the earlier terms of the sequence

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}$$

to see if this rule holds also for their denominators.

Fingers crossed, then.

I obviously cannot check the first fraction, $\frac{1}{1}$.

Why not?

Because there is no fraction before it.

A good point.

But the second fraction, $\frac{3}{2}$, has denominator 2, which is just $1 + 1$, the sum of the numerator 1 and denominator 1 of the first fraction $\frac{1}{1}$. This is getting exciting.

That's great. How about the third fraction $\frac{7}{5}$?

Right, Mr. $\frac{7}{5}$, let's see if you fit the theory. Your denominator is 5, is it not? Indeed it is, and the sum of the numerator and denominator of the previous fraction, $\frac{3}{2}$, is $3 + 2$, which I'm happy to say is none other than 5. This is fantastic! Who would have thought?

Great again! Now is there an equally simple rule for the numerators?

I hope so, because discovering that rule for the denominators gave me a great thrill.

We couldn't ask for more than that.

Right, back to the drawing board. So is there a connection between the numerator 17 of the fraction $\frac{17}{12}$ and the numbers 7 and 5 from the previous fraction $\frac{7}{5}$?

It would be marvelous if there were.

If I'm not mistaken, there is. It's simply that $17 = 7 + (2 \times 5)$.

Well spotted, though not quite as simple as the rule for the denominators.

No, but still easy enough.

Once you see it. How do you interpret this rule?

Doesn't it say that the next numerator is obtained by adding the numerator of the previous fraction to twice the denominator of the previous fraction?

Indisputable. You had better check this rule on the other fractions.

It works for the fraction $\frac{3}{2}$ since $3 = 1 + (2 \times 1)$, and it also works for $\frac{7}{5}$ since $7 = 3 + (2 \times 2)$.

This is wonderful. So how would you summarize the overall rule, which allows one to go from one fraction to the next?

Well, the general rule obtained by combining the denominator rule and the numerator rule seems to be:

> To get the denominator of the next fraction, add the numerator and denominator of the previous fraction; to get the numerator of the next fraction, add the numerator of the previous fraction to twice its denominator.

Well done! And a fairly straightforward rule, at that.

Isn't it amazing?

Indeed it is. After all, there was no reason to believe that there had to be any rule whatever connecting the fractions, but to find that there is one and that it's simple is remarkable.

I must now apply this general rule to $\frac{17}{12}$ to see what fraction comes out and to see if it has the property that its top squared minus twice the bottom squared is either 1 or -1.

Let's hope that the property holds.

According to the rule, the next fraction has a denominator of $17 + 12 = 29$ and a numerator of $17 + 2 \times 12 = 41$, and so is $\frac{41}{29}$.

Good. And now what are we hoping for?

Based on the pattern displayed by the previous four fractions, that $(41)^2 - 2(29)^2$ will work out to be -1.

Perform the acid test.

Here goes:

$$41^2 - 2(29)^2 = 1681 - 2(841) = 1681 - 1682 = -1$$

This is fantastic!

So now you have found another example of a perfect square that is within 1 of twice another perfect square—the whole point of this investigative detour—*without* having to go to the bother of extending your original tables.

That's true. Our more thorough examination of the four cases we found seems to have paid off.

A little thought can save a lot of computing.

I know that I couldn't have spotted this example with my tables because they give only the first thirty perfect squares along with their doubles; but can we be sure that there is not an m value between 17 and 41 that gives a square that is within 1 of twice another perfect square?

An excellent question. At the moment we can't be sure without checking. However, if there is such an m value, then it corresponds to a fraction $\frac{m}{n}$ that doesn't fit in with the above rule. Of course, this doesn't exclude the possibility of there being such a value. However, if you check, you won't find any such value.

I must calculate the next fraction generated by the rule to see if it also satisfies the plus or minus 1 property, to give it a name. Applying the rule to $\frac{41}{29}$ gives $41 + 29 = 77$ as the next denominator and $47 + (2 \times 29) = 99$ as the numerator.

So $\frac{99}{70}$ is the next fraction to be tested.

I predict that $m^2 - 2n^2 = 1$ in this case. The calculation

$$99^2 - 2(70)^2 = 9801 - 2(4900) = 9801 - 9800 = 1$$

verifies this. Great!

Bravo! What now?

Obviously, we can apply the rule over and over again and so generate an infinite sequence beginning with

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$$