# Mathematics for Computer Graphics

John Vince

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With 175 Illustrations

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**Dedication**

I dedicate this book to my wife Annie, who has had to tolerate a year of me reading math books in bed, on planes, boats, trains, in hotels, in the garden, in the bath, on holiday, and probably in my sleep!

# Contents

# Preface

Mathematics is a beautiful subject. Its symbols, notation and abstract structures permit us to define, manipulate and resolve extremely complex problems. The symbols by themselves, however, are meaningless – they are nothing more than a calligraphic representation of a mental idea. If one does not understand such symbols, then the encoded idea remains a secret.

Having spent most of my life using mathematics, I am still conscious of the fact that I do not understand much of the notation used by mathematicians. And even when I feel that I understand a type of notation, I still ask myself "Do I *really* understand its meaning?". For instance, I originally studied to be an electrical engineer and was very familiar with $i = \sqrt{-1}$, especially when used to represent out of phase voltages and currents. I can manipulate complex numbers with some confidence, but I must admit that I do not understand the meaning of $i^i$. This hole in my knowledge makes me feel uncomfortable, but I suppose it is reassuring to learn that some of our greatest mathematicians have had problems understanding some of their own inventions.

Some people working in computer graphics have had a rigorous grounding in mathematics and can exploit its power to solve their problems. However, in my experience, the majority of people have had to pick up their mathematical skills on an *ad hoc* basis depending on the problem at hand. They probably had no intention of being mathematicians, nevertheless they still need to learn about the mathematics and apply it intelligently, which is where this book comes in.

To begin with, this book is not for mathematicians. They would probably raise their hands in horror about the lack of mathematical rigour I have employed, or probably not employed! This book is for people working in computer graphics who know that they have to use mathematics in their day-to-day work, and don't want to get too embroiled in axioms, truths and Platonic realities.

The book originally appeared as part of Springer's excellent "*Essential*" series, and this new revised edition includes an extended chapter on Analytical Geometry and extra chapters on Barycentric Coordinates and Worked Examples. The chapter on Barycentric Coordinates forced me to return to one of my favourite books *A vector Space Approach to Geometry* by Melvin Hausner. This contains a wonderful explanation of balancing masses and how the results lead to barycentric coordinates. It also illustrates how area and volume are a natural feature of vectors. The chapter on Worked Examples draws upon some material from my recent book *Geometry for Computer Graphics*.

Whilst writing this book I have borne in mind what it was like for me when I was studying different areas of mathematics for the first time. In spite of reading and rereading an explanation several times it could take days before "the penny dropped" and a concept became apparent. Hopefully, the reader will find the following explanations useful in developing their understanding of these specific areas of mathematics.

John Vince
Ringwood

# 1

# Mathematics

When I was taught mathematics at junior school in the late 1950s, there were no computers or calculators. Calculations, whether they were addition, subtraction, multiplication, division or square roots, had to be worked out in one's head or with pencil and paper. We learnt our 'times tables' by reciting them over and over again until we could give the product of any pair of numbers up to 12 – numbers higher than 12 were computed long hand.

I was fortunate in having a teacher who appreciated the importance of mathematics, and without knowing it at the time, I began a journey into a subject area that would eventually bring my knowledge of mathematics to life in computer graphics.

Today, students have access to calculators that are virtually miniature computers. They are programmable and can even display graphs on small LCD screens. Unfortunately, the policy pursued by some schools has ensured that generations of children are unable to compute simple arithmetic operations without the aid of a calculator. I believe that such children have been disadvantaged, as they are unable to visualize the various patterns that exist in numbers such as odd numbers $(1, 3, 5, 7, \ldots)$, even numbers $(2, 4, 6, 8, \ldots)$, prime numbers $(2, 3, 5, 7, 11, \ldots)$, squares $(1, 4, 9, 16, 25, \ldots)$ and Fibonacci numbers $(0, 1, 1, 2, 3, 5, 8, \ldots)$. They will not know that it is possible to multiply a two-digit number, such as 17, by 11, simply by adding 1 to 7 and placing the result in the middle to make 187.

Although I do appreciate the benefits of calculators, I believe that they are introduced into the curriculum far too early. Children should be given the opportunity to develop a sense of number, and the possibility of developing a love for mathematics, before they discover the tempting features of a digital calculator.

'I am no good at mathematics' is a common response from most people when asked about their mathematical abilities. Some suggest that their brain is unable to cope with numbers, some claim that it's boring, and others put it down to inadequate teaching. Personally, I am not very good at mathematics, but I delight in reading books about mathematicians and the history of mathematics, and applying mathematics to solve problems in computer graphics. I am easily baffled by pages of abstract mathematical symbols, but readily understand the application of mathematics in a practical context.

It was only when I started programming computers to produce drawings and pictures that I really appreciated the usefulness of mathematics. Multiplication became synonymous with scaling; division created perspective; sines and cosines rotated objects; tangents produced shearing, and geometry and trigonometry provided the analytical tools to solve all sorts of other problems. Such a toolkit is readily understood and remembered.

## 1.1 Is Mathematics Difficult?

'Is mathematics difficult?' I suppose that there is no real answer to this question, because it all depends upon what we mean by 'mathematics' and 'difficult'. But if the question is rephrased slightly: 'Is the mathematics of computer graphics difficult?' then the answer is a definite no. What's more, I believe that the subject of computer graphics can instill in someone a love for mathematics. Perhaps 'love' is too strong a word, but I am convinced that it is possible to 'make friends' with mathematics.

For me, mathematics should be treated like a foreign language: You only need to learn an appropriate vocabulary to survive while visiting another country. If you attempt to memorize an extended vocabulary, and do not put it into practice, it is highly likely that you will forget it. Mathematics is the same. I know that if I attempted to memorize some obscure branch of mathematics, such as vector calculus, I would forget it within days if I did not put it to some practical use.

Fortunately, the mathematics needed for computer graphics is reasonably simple and covers only a few branches such as algebra, trigonometry, vectors, geometry, transforms, interpolation, curves and patches. Although these topics do have an advanced side to them, in most applications we only need to explore their intermediate levels.

## 1.2 Who should Read this Book?

I have written this book as a reference for anyone intending to study topics such as computer graphics, computer animation, computer games or virtual reality, especially for people who want to understand the technical aspects.

Although it is possible to study these topics without requiring the support of mathematics, increasingly, there are situations and projects that require animators, programmers and technical directors to resort to mathematics to resolve unforeseen technical problems. This may be in the form of a script or an extra piece of program code.

## 1.3 Aims and Objectives of this Book

One of the aims of this book is to bring together a range of useful mathematical topics that are relevant to computer graphics. And the real objective is to provide programmers and animators with an understanding of mathematics so that they can solve all sorts of problems with confidence.

I have attempted to do this by exploring a range of mathematical topics, without intimidating the reader with mathematical symbols and abstract ideas. Hopefully, I will be able to explain each topic in a simple and practical manner, with a variety of practical examples.

This is far from being an exhaustive study of the mathematics associated with computer graphics. Each chapter introduces the reader to a new topic, and should leave the reader confident and capable of studying more advanced books.

## 1.4 Assumptions Made in this Book

I suppose that I do expect that readers will have some understanding of arithmetic and a general knowledge of the principles of mathematics, such as the ideas of algebra. But, apart from that, each subject will be introduced as though it were the first time it had been discovered.

In the chapter on curves and surfaces I have used a little calculus. Readers who have not studied this subject should not be concerned about missing some vital piece of information. I only included it to keep the explanation complete.

## 1.5 How to Use the Book

I would advise starting at the beginning and proceeding chapter by chapter. Where a subject seems familiar, just jump ahead until a challenge is discovered. Once you have read the book, keep it handy so that you can refer to it when the occasion arises.

Although I have tried to maintain a sequence to the mathematical ideas, so that one idea leads to another, in some cases this has proved impossible. For example, determinants are referred to in the chapter on vectors, but they are described in detail in the next chapter on transforms. Similarly, the later chapter on analytic geometry contains some basic ideas of geometry, but its position was dictated by its use of vectors. Consequently, on some occasions, the reader will have to move between chapters to read about related topics.

# 2

# Numbers

All sorts of number system have been proposed by previous civilizations, but our current system is a positional number system using a base 10. The number 1234 really means the sum of one thousand, plus two hundreds, plus three tens, plus four ones, which can be expressed as $1 \times 1000 + 2 \times 100 + 3 \times 10 + 4 \times 1$. It should be obvious that the base 10 is nothing special, it just so happens that human beings have evolved with 10 digits, which we use for counting. This suggests that any number can be used as a base: 2, 3, 4, 5, 6, 7, etc. In fact, the decimal number system is not very convenient for computer technology, where electronic circuits switch on and off billions of times a second using binary numbers – numbers to a base 2 – with great ease. In this text there is no real need to explore such numbers. This can be left to programmers who have to master number systems such as binary (base 2), octal (base 8) and hexadecimal (base 16).

The only features of numbers we have to revise in this chapter are the families of numbers that exist, what they are used for, and any problems that arise when they are stored in a computer. Let's begin with the natural numbers.

## 2.1 Natural Numbers

The *natural numbers* $\{0, 1, 2, 3, 4, \ldots\}$ are used for counting, ordering and labelling. Note that negative numbers are not included. We often use natural numbers to subscript a quantity to distinguish one element from another, e.g. $x_1, x_2, x_3, x_4, \ldots$

## 2.2 Prime Numbers

A natural number that can be divided only by 1 and itself, without leaving a remainder, is called a *prime number*. Examples are $\{2, 3, 5, 7, 11, 13, 17\}$. There are 25 primes less than 100, 168 primes less than 1000 and 455 052 512 primes less than 10 000 000 000. The *fundamental theory of arithmetic* states, 'Any positive integer (other than 1) can be written as the product of prime numbers in one and only one way.' For example, $25 = 5 \times 5; 26 = 2 \times 13; 27 = 3 \times 3 \times 3; 28 = 2 \times 2 \times 7; 29 = 29; 30 = 2 \times 3 \times 5$ and $92\,365 = 5 \times 7 \times 7 \times 13 \times 29$.

In 1742 Christian Goldbach conjectured that every even integer greater than 2 could be written as the sum of two primes:

$$\begin{aligned} 4 &= 2 + 2 \\ 14 &= 11 + 3 \\ 18 &= 11 + 7, \text{etc.} \end{aligned}$$

No one has ever found an exception to this conjecture, and no one has ever proved it.

Although prime numbers are enigmatic and have taxed the brains of the greatest mathematicians, unfortunately they play no part in computer graphics!

## 2.3 Integers

*Integers* include negative numbers, as follows: $\{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

## 2.4 Rational Numbers

*Rational* or *fractional* numbers are numbers that can be represented as a fraction. For example, $2, \sqrt{16}, 0.25$ are rational numbers because

$$2 = \frac{4}{2}, \quad \sqrt{16} = 4 = \frac{8}{2}, \quad 0.25 = \frac{1}{4}$$

Some rational numbers can be stored accurately inside a computer, but many others can only be stored approximately. For example, $4/3 = 1.333\,333\ldots$ produces an infinite sequence of threes and has to be truncated when stored as a binary number.

## 2.5 Irrational Numbers

*Irrational numbers* cannot be represented as fractions. Examples are $\sqrt{2} = 1.414\,213\,562\ldots, \pi = 3.141\,592\,65\ldots$ and $\mathrm{e} = 2.718\,281\,828\ldots$ Such numbers

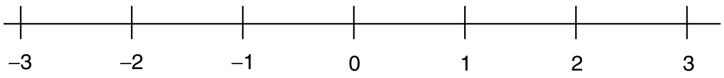


**Fig. 2.1.** The number line.

never terminate and are always subject to a small error when stored within a computer.

## 2.6 Real Numbers

Rational and irrational numbers together comprise the *real numbers*.

## 2.7 The Number Line

It is convenient to organize numbers in the form of an axis to give them a spatial significance. Figure 2.1 shows such a *number line*, which forms an axis as used in graphs and coordinate systems. The number line also helps us understand complex numbers, which are the 'king' of all numbers.

## 2.8 Complex Numbers

Leonhard Euler (1707–1783) (whose name rhymes with *boiler*) played a significant role in putting *complex numbers* on the map. His ideas on rotations are also used in computer graphics to locate objects and virtual cameras in space, as we shall see later on.

Complex numbers resolve some awkward problems that arise when we attempt to solve certain types of equations. For example, $x^2 - 4 = 0$ has solutions $x = \pm 2$. But $x^2 + 4 = 0$ has no obvious solutions using real or integer numbers. However, the number line provides a graphical interpretation for a new type of number, the complex number. The name is rather misleading: it is not complex, it is rather simple.

Consider the scenario depicted in Figure 2.2. Any number on the number line is related to the same number with the opposite sign via an anti-clockwise rotation of 180°. For example, if 3 is rotated 180° about zero it becomes $-3$, and if $-2$ is rotated 180° about zero it becomes 2.

We can now write $-3 = (-1) \times 3$, or $2 = (-1) \times -2$, where $-1$ is effectively a rotation through 180°. But a rotation of 180° can be interpreted as two consecutive rotations of 90°, and the question now arises: What does a rotation of 90° signify? Well, let's assume that we don't know what the answer is going to be – even though some of you do – we can at least give a name to the operation, and what better name to use than i.

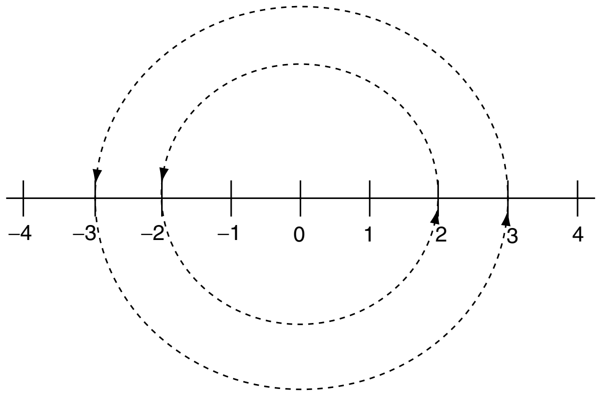


**Fig. 2.2.** Rotating numbers through 180° reverses their sign.

So the letter i represents an anticlockwise rotation of 90°. Therefore i2 is equivalent to lifting 2 out of the number line, rotating it 90° and leaving it hanging in limbo. But if we take this '*imaginary*' number and subject it to a further 90° rotation, i.e. ii2, it becomes −2. Therefore, we can write ii2 = −2, which means that ii = −1. But if this is so, $\mathrm{i} = \sqrt{-1}$!

This gives rise to two types of number: real numbers and complex numbers. Real numbers are the everyday numbers we use for counting and so on, whereas complex numbers have a mixture of real and imaginary components, and help resolve a wide range of mathematical problems.

Figure 2.3 shows how complex numbers are represented: the horizontal number line represents the *real component*, and the vertical number line represents the *imaginary component*.

For example, the complex number $P(1 + \mathrm{i}2)$ in Figure 2.3 can be rotated 90° to $Q$ by multiplying it by i. However, we must remember that ii = −1:

$$\begin{aligned}\mathrm{i}(1+\mathrm{i}2) &= \mathrm{i}1 + \mathrm{ii}2\\ &= \mathrm{i}1 - 2\\ &= -2 + \mathrm{i}1\end{aligned}$$

$Q(-2 + \mathrm{i}1)$ can be rotated another 90° to $R$ by multiplying it by i:

$$\begin{aligned}\mathrm{i}(-2+\mathrm{i}1) &= \mathrm{i}(-2) + \mathrm{ii}1\\ &= -\mathrm{i}2 - 1\\ &= -1 - \mathrm{i}2\end{aligned}$$

$R(-1 - \mathrm{i}2)$ in turn, can be rotated 90° to $S$ by multiplying it by i:

$$\begin{aligned}\mathrm{i}(-1-\mathrm{i}2) &= \mathrm{i}(-1) - \mathrm{ii}2\\ &= -\mathrm{i}1 + 2\\ &= 2 - \mathrm{i}1\end{aligned}$$

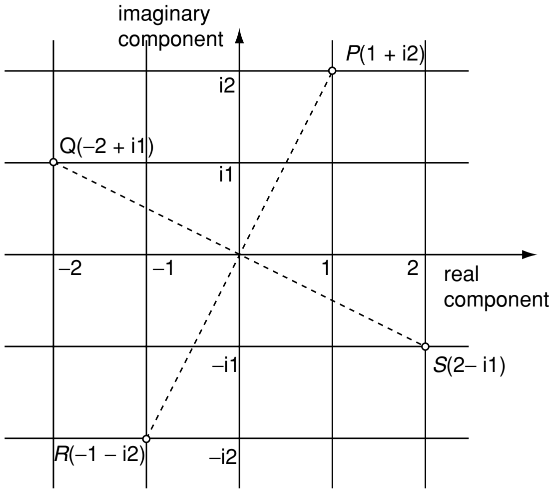


**Fig. 2.3.** The graphical representation of complex numbers.

Finally, $S(2 - \mathrm{i}1)$ can be rotated 90° to $P$ by multiplying it by i:

$$\begin{aligned} \mathrm{i}(2 - \mathrm{i}1) &= \mathrm{i}2 - \mathrm{ii}1 \\ &= \mathrm{i}2 + 1 \\ &= 1 + \mathrm{i}2 \end{aligned}$$

Although we rarely use complex numbers in computer graphics, we can see that they are intimately related to Cartesian coordinates, and that the ordered pair $(x, y) \equiv x + \mathrm{i}y$.

Before concluding this chapter, I cannot fail to include the famous equation discovered by Euler:

$$\mathrm{e}^{\mathrm{i}\pi} + 1 = 0 \tag{2.1}$$

which integrates 0, 1, e, $\pi$ and i in a simple and beautiful arrangement, and is on a par with Einstein's $e = mc^2$.

## 2.9 Summary

This short chapter made sure that the terminology of numbers was understood, and now provides a good link into the basics of algebra.

# 3

# Algebra

This chapter reviews the basic elements of algebra to prepare the reader for the algebraic manipulations used in later chapters. Although algebra can be a very abstract mathematical tool, here we only need to explore those practical features relevant to its application to computer graphics.

## 3.1 Notation

The word '*algebra*' comes from the Arabic *al-jabr w'al-muqabal*, meaning 'restoration and reduction'. Today's algebraic notation has evolved over thousands of years during which different civilizations have developed ways of annotating mathematical and logical problems. In retrospect, it does seem strange that centuries passed before the 'equals' sign (=) was invented and concepts such as 'zero' (CE 876) were introduced, especially as they now seem so important. But we are not at the end of this evolution, because new forms of annotation and manipulation will continue to emerge as new mathematical ideas are invented.

One fundamental concept of algebra is the idea of giving a name to an unknown quantity. For example, $m$ is often used to represent the slope of a 2D line, and $c$ is the line's *y-coordinate* where it intersects the $y$-axis. René Descartes (1596–1650) formalized the idea of using letters from the beginning of the alphabet ($a, b, c$, etc.) to represent arbitrary numbers, and letters at the end of the alphabet ($p, q, r, s, t, \dots x, y, z$) to identify variables representing quantities such as pressure ($p$), temperature ($t$), and coordinates ($x, y, z$).

With the aid of the basic arithmetic operators $+, -, \times, \div$ we can develop expressions that describe the behaviour of a physical process or a specific computation. For example, the expression $ax+by-d$ equals zero for a straight line. The variables $x$ and $y$ are the coordinates of any point on the line and the values of $a, b, d$ determine the position and orientation of the line. There is an implied multiplication between $ax$ and $by$, which would be expressed as $a^*x$ and $b^*y$ if we were using a programming language.

The = sign permits the line equation to be expressed as a self-evident statement: $0 = ax + by - d$. Such a statement implies that the expressions on the left- and right-hand sides of the = sign are 'equal' or 'balanced'. So whatever is done to one side must also be done to the other in order to maintain equality or balance. For example, if we add $d$ to both sides, the straight-line equation becomes $d = ax + by$. Similarly, we could double or treble both expressions, divide them by 4, or add 6, without disturbing the underlying relationship.

Algebraic expressions also contain a wide variety of other notation, such as

| | |
|---|---|
| $\sqrt{\mathrm{x}}$ | square root of $x$ |
| $\sqrt[n]{\mathrm{x}}$ | $n$th root of $x$ |
| $x^n$ | $x$ to the power $n$ |
| $\sin(x)$ | sine of $x$ |
| $\cos(x)$ | cosine of $x$ |
| $\tan(x)$ | tangent of $x$ |
| $\log(x)$ | logarithm of $x$ |
| $\ln(x)$ | natural logarithm of $x$ |

Parentheses are used to isolate part of an expression in order to select a sub-expression that is manipulated in a particular way. For example, the parentheses in $c(a+b)+d$ ensure that the variables a and $b$ are added together before being multiplied by $c$ and finally added to d.

## 3.2 Algebraic Laws

There are three basic laws that are fundamental to manipulating algebraic expressions: associative, commutative and distributive. In the following descriptions, the term *binary operation* represents the arithmetic operations $+$, $-$ or $\times$, which are always associated with a pair of numbers or variables.

### 3.2.1 Associative Law

The *associative law* in algebra states that when three or more elements are linked together through a binary operation, the result is independent of how each pair of elements is grouped. The associative law of addition is

$$a + (b + c) = (a + b) + c \tag{3.1}$$

e.g. $1 + (2 + 3) = (1 + 2) + 3$

and the associative law of multiplication is

$$a \times (b \times c) = (a \times b) \times c \tag{3.2}$$

e.g. $1 \times (2 \times 3) = (1 \times 2) \times 3$
Note that substraction is not associative:

$$a - (b - c) \neq (a - b) - c \tag{3.3}$$

e.g. $1 - (2 - 3) \neq (1 - 2) - 3$

### 3.2.2 Commutative Law

The *commutative law* in algebra states that when two elements are linked through some binary operation, the result is independent of the order of the elements. The commutative law of addition is

$$a + b = b + a \tag{3.4}$$

e.g. $1 + 2 = 2 + 1$
and the commutative law of multiplication is

$$a \times b = b \times a \tag{3.5}$$

e.g. $2 \times 3 = 3 \times 2$
Note that subtraction is not commutative:

$$a - b \neq b - a \tag{3.6}$$

e.g. $2 - 3 \neq 3 - 2$

### 3.2.3 Distributive Law

The *distributive law* in algebra describes an operation which when performed on a combination of elements is the same as performing the operation on the individual elements. The distributive law does not work in all cases of arithmetic. For example, multiplication over addition holds:

$$a \times (b + c) = ab + ac \tag{3.7}$$

e.g. $3 \times (4 + 5) = 3 \times 4 + 3 \times 5$
whereas addition over multiplication does not:

$$a + (b \times c) \neq (a + b) \times (a + c) \tag{3.8}$$

e.g. $3 + (4 \times 5) \neq (3 + 4) \times (3 + 5)$

Although most of these laws seem to be natural for numbers, they do not necessarily apply to all mathematical constructs. For instance, the *vector product*, which multiplies two vectors together, is not commutative.

## 3.3 Solving the Roots of a Quadratic Equation

To put the above laws and notation into practice, let's take a simple example to illustrate the logical steps in solving a problem. The task involves solving the roots of a quadratic equation, i.e. those values of $x$ that make the equation equal zero.

Given the quadratic equation where $a \neq 0$:

$$ax^2 + bx + c = 0$$

*Step 1*: subtract $c$ from both sides:

$$ax^2 + bx = -c$$

*Step 2*: divide both sides by $a$:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

*Step 3*: add $\frac{b^2}{4a^2}$ to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

*Step 4*: factorize the left side:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

*Step 5*: make $4a^2$ the common denominator for the right side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

*Step 6*: take the square root of both sides:

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

*Step 7*: subtract $\frac{b}{2a}$ from both sides:

$$x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

*Step 8*: rearrange the right side:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (3.9)$$

This last expression gives the roots for any quadratic equation.