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André Voros

Zeta Functions over Zeros of Zeta Functions

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André Voros
CEA-Saclay
Institut de Physique Théorique (IPhT)
Orme des Merisiers
91191 Gif-sur-Yvette
France
andre.voros@cea.fr

Appendix D:
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To Estelle
To Magali and Sandrine

Preface

In the Riemann zeta function $\zeta(s)$, the non-real zeros or *Riemann zeros*, denoted ρ, play an essential role mainly in number theory, and thereby generate considerable interest. However, they are very elusive objects. Thus, no individual zero has an analytically known location; and the *Riemann Hypothesis*, which states that all those zeros should lie on the *critical line*, i.e., have real part $\frac{1}{2}$, has challenged mathematicians since 1859 (exactly 150 years ago).

For analogous symmetric sets of numbers $\{v_k\}$, such as the roots of a polynomial, the eigenvalues of a finite or infinite matrix, etc., it is well known that *symmetric functions* of the $\{v_k\}$ tend to have more accessible properties than the individual elements v_k. And, we find the largest wealth of explicit properties to occur in the *(generalized) zeta functions* of the generic form

$$\mathrm{Zeta}(s,a) = \textstyle\sum_k (v_k + a)^{-s}$$

(with the extra option of replacing v_k here by selected functions $f(v_k)$).

Not surprisingly, then, zeta functions over the Riemann zeros have been considered, some as early as 1917. What is surprising is *how small* the literature on those zeta functions has remained overall. We were able to spot them in barely a dozen research articles over the whole twentieth century and in none of the *books* featuring the Riemann zeta function. So the domain exists, but it has remained largely confidential and sporadically covered, in spite of a recent surge of interest.

Could it then be that those zeta functions have few or uninteresting properties? In actual fact, their study yields an abundance of quite explicit results. The significance or usefulness of the latter may then be questioned: at this moment, we can only answer that regarding the Riemann zeros, any explicit result, even of a collective nature, is of *potential* value. Hence we may turn over the idea that zeta functions over the Riemann zeros have stagnated because they were not so interesting: it could also be that those functions have lagged behind in their use simply because their properties never came to be fully displayed.

So, the primary aim of this monograph is to fill that very specific but definite *gap*, by offering a coherent and synthetic description of the zeta functions over the Riemann zeros (and immediate extensions thereof); we call them “superzeta” functions here for brevity. Modeled on special-function handbooks (our main reference case being the Hurwitz zeta function $\zeta(s,a)$), this book centers on delivering extensive lists of concrete explicit properties and tables of handy special-value formulae for superzeta functions, grouped in three core chapters plus Appendix B (for the variant case built over zeros of Selberg zeta functions). In that core, we mainly wish to provide readers, assuming they have specific queries about superzeta functions, with a broad panel of explicit answers. For such a purpose, the key contents of the book may be just Chap. 5 (for initial orientation) and the final results, including 20 tables of special-value formulae. The rest of the text is rather backstage material, showing justifications, perspective, and references for those end results.

For the reasons given above, we grade no individual result or formula as more or less “useful,” but place them all on an equal footing. Our main justification to date for tackling those superzeta functions is simply “Because they’re there” (like a famous mountaineer’s reply).

We now outline the contents.

Two introductory chapters review our main analytical techniques: miscellaneous notation and tools, specially the Mellin transformation (Chap. 1), and zeta-regularized products (Chap. 2). The next two chapters, still introductory, survey the Riemann zeta function itself (and close kin, the Dirichlet beta and Hurwitz zeta functions), so as to make the book reasonably self-contained and tutorial. All review sections are, however, filtered hierarchically: the aspects most central to us are exposed in detail, others more sketchily (and a few just get mentioned). We do not try to compete with the many exhaustive treatises on the Riemann zeta function; on the other hand, a shorter tutorial like ours might suit readers seeking to learn about that function from a purely *analytical*, as opposed to number-theoretical, angle.

The next two chapters begin to address the superzeta functions themselves: Chap. 5 gives an overview, and the following one introduces Explicit Formulae from number theory, which are then applied to superzeta functions (and compared to Selberg trace formulae for spectral zeta functions).

Chapters 7–10 form the core of the study: *three kinds* of superzeta functions are thoroughly described in the first three chapters, then extended to zeros of more general L- or zeta-functions in Chap. 10; except for Chap. 9, every core chapter (plus Appendix B for the Selberg case) culminates in detailed Tables of special-value formulae.

To close, Chap. 11 shows one application of a superzeta function: a recently obtained asymptotic criterion for the Riemann Hypothesis (based on the Keiper–Li coefficients used by *Li’s criterion*). Finally, four Appendices treat extra issues (A: some numerical aspects; B: extension to zeta functions over zeros of *Selberg* zeta functions; C: on $(\log|\zeta|)^{(2m+1)}(\frac{1}{2})$, etc.; D: an English translation of Mellin’s seminal 1917 paper in German).

As we aim to throw light on an unpublicized subject on which this is the very first book as far as we know, our text is kept concrete and expository through the first half at least, favoring elementary and economical techniques. Exercises are also proposed in the form of peripheral results left for the reader to derive. Our wish is to have built a compact reference guide, a kind of "Everything you always wanted to know about superzeta functions ..." handbook. For the sake of improvement, we gratefully welcome any error reports from readers (and will post errata as needed).

This text is thus directed at readers interested in analytical aspects of number theory. It ought to be accessible to mathematicians from the graduate level; its main assumed background is in analysis (real and complex: series and integrals, analytic and special functions, asymptotics).

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For this study, I am primarily indebted to Prof. P. Cartier who initiated our collaboration on trace formulae in the late 80s, ushering me into an area entirely new to me; I express to him my gratitude for his stimulating help and encouragement.

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My deepest thanks to my spouse Estelle, for her enduring patience, understanding, and support at all times but specially during the completion of the writing which seemed to stretch forever; this strain was also shared from a greater distance by our daughters Magali and Sandrine: my thanks go to them too.

Saclay, July 2009 *André Voros*

Contents

List of Special Symbols

Occasional, or inversely universal, symbols are not listed. (For us, sets like $\mathbb{N}$, $\mathbb{R}_+$ *include* 0, while $\mathbb{N}^*$, $\mathbb{R}_+^*$, etc., exclude it.) Appendix D is not included.

$\mathcal{A}_S$	Area of a surface S
a	Parity bit (0 or 1) for a Dirichlet character
B	$(\log \mathit{\Xi})'(0)$ for the completed Riemann zeta function $\mathit{\Xi}(\cdot)$
B_n	Bernoulli numbers
$B_n(\cdot)$	Bernoulli polynomials
$\mathbf{D}(x)$	Zeta-regularized form of trivial factor $\mathbf{G}(x)$
$\mathscr{D}(x)$	Zeta-regularized form of completed zeta- or L-function $\mathit{\Xi}(x)$
$\mathcal{D}(v)$	Zeta-regularized form of $\mathit{\Xi}(\frac{1}{2} + v^{1/2})$
d	Period ("modulus") of a Dirichlet character
d_K	Discriminant of an algebraic number field K
E_n	Euler numbers
$\mathrm{FP}_{x=x_0} f$	Finite part of a function $f(x)$ at x_0
g	Genus (for a surface)
g_n	Stieltjes constants (in our normalization), *see* γ_{n-1}
g_n^{c}	Stieltjes cumulants (in our normalization), *see* $\tilde{\gamma}_{n-1}$
$\mathbf{G}(\cdot)$	Trivial (entire) factor in a zeta- or L-function
H_n	Harmonic numbers
$K_\nu(\cdot)$	Modified Bessel function
$L(\cdot)$	Generic ("primary") zeta- or L-function
$L_\chi(\cdot)$	L-function for a Dirichlet character χ
ℓ_ϖ	Length of a periodic geodesic ϖ
$\mathrm{M}f$	Mellin transform of a function f
$N(T)$	Counting function of (e.g., Riemann) zeros' ordinates
$\overline{N}(T)$	Trivial-factor contribution to $N(T)$
$\overline{N}_0(T)$	Asymptotic form of $N(T)$ (mod $\mathrm{O}(\log T)$)
n_K	Degree of an algebraic number field K
$\{p\}$	Set of prime numbers
RH	Riemann Hypothesis
$\mathcal{R}_m$	Residue of $\mathcal{Z}_0(\sigma)$ at the pole $\sigma = \frac{1}{2} - m$
$\mathcal{R}_m(t)$	Residue of $\mathcal{Z}(\sigma \mid t)$ at the pole $\sigma = \frac{1}{2} - m$

$S(T)$	Contribution of $\zeta(x)$ to the function $N(T)$ for the Riemann zeros
$\{u_k\}$	Sequence $\{\rho\,(1-\rho) \mid \operatorname{Im}\,\rho > 0\}$ over the nontrivial zeros
$V(\cdot)$	Cramér's function
$Z(\cdot)$	Generic Zeta-type function
$Z(\cdot,\cdot)$	Generic generalized zeta function
$Z'(\cdot,\cdot)$	Derivative of $Z(\cdot,\cdot)$ in the *first* argument (the exponent)
$\mathbf{Z}(s\,\vert\,t)$	Generalized zeta function over the trivial zeros of a zeta function
$\mathscr{Z}(s\,\vert\,t)$	Superzeta function of first kind
$\mathscr{Z}_0(s)$	$\mathscr{Z}(s\,\vert\,t=0)$
$\mathscr{Z}_*(s)$	$\mathscr{Z}(s\,\vert\,t=\frac{1}{2})$
$\mathcal{Z}(\sigma\,\vert\,t)$	Superzeta function of second kind
$\mathcal{Z}_0(\sigma)$	$\mathcal{Z}(\sigma\,\vert\,t=0)$
$\mathcal{Z}_*(\sigma)$	$\mathcal{Z}(\sigma\,\vert\,t=\frac{1}{2})$
$\mathfrak{Z}(s\,\vert\,\tau)$	Superzeta function of third kind
$\beta(\cdot)$	Dirichlet beta-function
$\Gamma(\cdot)$	Euler Gamma function
γ	Euler's constant
γ_j	Stieltjes constants, *see* our modified notation g_{j+1}
$\tilde\gamma_j = \eta_j$	Stieltjes "cumulants", *see* our modified notation g^{c}_{j+1}
$\Delta(\cdot)$	Generic Hadamard product
$\Delta_0(\cdot)$	Generic Weierstrass product
$\Delta_\infty(\cdot)$	Generic zeta-regularized product
δ_1	Discrepancy in $\mathscr{Z}(s\,\vert\,t)$ at $s=1$
$\delta_{j,k}$	Kronecker delta
$\zeta(\cdot)$	Riemann zeta function
$\zeta(\cdot,\cdot)$	Hurwitz zeta function
$\zeta_K(\cdot)$	Dedekind zeta function for an algebraic number field K
$\boldsymbol{\zeta}_S(\cdot)$	Selberg zeta function for a hyperbolic surface S
η_j	Stieltjes "cumulants", *see* $\tilde\gamma_j$
$\Theta(\cdot)$	Theta-type function
κ_k	Wavenumbers ([eigenvalues$-\frac{1}{4}]^{1/2}$) for a hyperbolic 2D Laplacian
$\Lambda(n)$	von Mangoldt function
λ_n	Keiper–Li coefficients
μ_0	Order (of a sequence, of an entire function)
$\Xi(\cdot)$	Completed zeta- or L-function
$\{\varpi\}$	Set of primitive oriented periodic geodesics on a surface
$\{\rho\}$	Set of nontrivial zeros of a zeta- or L-function (e.g., Riemann)
$\{\tau_k\}$	Sequence $\{\mathrm{i}^{-1}(\rho-\frac{1}{2}) \mid \operatorname{Im}\,\rho > 0\}$ over the nontrivial zeros
χ	Generic Dirichlet character
χ_{d_K}	Kronecker symbol for the discriminant d_K
$\psi(\cdot)$	Digamma function $[\Gamma'/\Gamma](\cdot)$
$\Omega_1,\ \Omega_2$	t-domains for superzeta functions of first, second kind

Chapter 1
Introduction

1.1 Symmetric Functions

The non-real zeros of the Riemann zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-s} \qquad (\text{Re } s > 1), \tag{1.1}$$

called the *Riemann zeros* and usually denoted ρ, are most elusive quantities. Thus, no individual Riemann zero is analytically known; and the Riemann Hypothesis (RH): Re $\rho = \frac{1}{2}$ $(\forall\rho)$, has stayed unresolved since 1859 [92].

For analogous finite or infinite sets of numbers $\{v_k\}$, like the roots of a polynomial, the eigenvalues of a matrix, or the discrete spectrum of a linear operator, the *symmetric functions* of $\{v_k\}$ tend to be much more accessible. Some common *types* of additive symmetric functions, to be denoted Theta, Zeta and (log Delta) here, are formally given by

$$\begin{aligned}
\text{Theta}(z) &\stackrel{\text{def}}{=} \textstyle\sum_k \mathrm{e}^{-zx_k}, \\
\text{Zeta}(s) &\stackrel{\text{def}}{=} \textstyle\sum_k x_k^{-s}, \\
\text{Delta}(a) &\stackrel{\text{def}}{=} \textstyle\prod_k (x_k + a) \qquad \text{or, if this diverges,} \\
(\log \text{Delta})^{(m)}(a) &\stackrel{\text{def}}{=} (-1)^{m-1}(m-1)! \textstyle\sum_k (x_k + a)^{-m} \quad \text{for some } m \geq 1,
\end{aligned}$$

where $x_k = v_k$ or some other function $f(v_k)$ (such that no $x_k = 0$ and, e.g., Re $x_k \to +\infty$). It is useful to allow at least the a-shift in this remapping, thereby obtaining a two-variable or generalized zeta function, in analogy with the Hurwitz function $\zeta(s, a) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (k + a)^{-s}$:

$$\text{Zeta}(s, a) \stackrel{\text{def}}{=} \textstyle\sum_k (x_k + a)^{-s}. \tag{1.2}$$

Here we think of s as the main argument, the variable in which analyticity properties and special values are studied, and of a as an (auxiliary) shift

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parameter which adds flexibility; i.e., we view $\mathrm{Zeta}(s,a)$ as a parametric family in the type $\mathrm{Zeta}(s)$ (accordingly denoting $\mathrm{Zeta}'(s,a) \stackrel{\mathrm{def}}{=} \partial_s \mathrm{Zeta}(s,a)$).

The gain with $\mathrm{Zeta}(s,a)$ is that it encompasses the last three types above:

$$\mathrm{Zeta}(s) = \mathrm{Zeta}(s,0),$$

$$\mathrm{Delta}(a) \stackrel{\mathrm{def}}{=} \exp[-\mathrm{Zeta}'(0,a)]$$

(when Delta is an *infinite* product, this is zeta-regularization, see Chap. 2),

$$(\log \mathrm{Delta})^{(m)}(a) = (-1)^{m-1}(m-1)!\, \mathrm{Zeta}(m,a);$$

while the Zeta type is simply a Mellin transform of the Theta type, as

$$\mathrm{Zeta}(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Theta}(z)\, \mathrm{e}^{-za}\, z^{s-1} \mathrm{d}z.$$

So, formally, all those types of symmetric functions look interchangeable and their properties convertible from one to the other. However, experience (especially from spectral theory) tells that zeta functions are those which display the most explicit properties, reaching to computable special values (values at integers) as in the case of $\zeta(s)$ itself.

Again from spectral techniques we borrow the idea that, besides the above shift operation, *nonlinear* remappings $x_k = f(v_k)$ may prove suitable before building the symmetric functions. For instance, if $\{v_k\}$ is the spectrum of a Laplacian on a manifold, both choices $x_k = v_k$ and $x_k = \sqrt{v_k}$ have their own merits: on the circle, with the spectrum $\{n^2\}_{n\in\mathbb{Z}}$, the resulting Theta-type functions are, respectively, a Jacobi θ function and $\coth z/2$, a generating function for the Poisson summation formula as in (1.13); whereas on a compact hyperbolic surface (normalized to curvature -1), an even better choice than $\sqrt{v_k}$ is $x_k = (v_k - \frac{1}{4})^{1/2}$, as the Selberg trace formula shows. (This formula expresses additive symmetric functions of precisely *the latter* x_k as dual sums carried by the periodic geodesics of the surface, see Sect. 6.3.1.)

It is then very natural to study symmetric functions of the Riemann zeros in a similar manner, and this has happened. Indeed,

- Some zeta functions built over the Riemann zeros have appeared in a few works, as early as 1917
- A universal tool exists to evaluate fairly general additive symmetric functions of the Riemann zeros: the Guinand–Weil "Explicit Formulae."

Still, we feel that our subject (zeta functions over the Riemann zeros) remains far from exhausted. For one thing, the existing studies are surprisingly few over a long stretch of time; they are neither systematic nor error-free, are often unaware of one another, and none has made it to the classic textbooks on $\zeta(s)$; consequently there is no comprehensive, easily accessible treatment of zeta functions over the Riemann zeros. Calculations in this field continue to appear on a case-by-case basis.

Neither do the classic Explicit Formulae settle the issue of these zeta functions as mere special instances: on the contrary, the most interesting particular zeta functions over the Riemann zeros lie outside the standard range of validity (i.e., of convergence) of those formulae.

In contrast, a dedicated study of these zeta functions uncovers a wealth of explicit results, many of which were not given or even suspected in the literature. The question of the importance or usefulness of those results will not be addressed: the answer may lie in an undefined future.

There is no standardized terminology for zeta functions over zeros of zeta functions. Chakravarty [17] used the name "secondary zeta functions," but to denote several Dirichlet series apart from $\zeta(x)$ itself which is the "primary" zeta function (that which supplies its zeros). Here, to have a short and specific name, we choose to call "superzeta" functions all second-generation zeta functions built over zeros of other, "primary", zeta functions.

We continue this introduction with some essential notation, then we will recall the most basic tools that will often serve later.

1.2 Essential Basic Notation

As a rule, we refer to [1, 33].

Bernoulli polynomials (definition by generating function):

$$\frac{z\,\mathrm{e}^{zy}}{\mathrm{e}^z - 1} \equiv \sum_{n=0}^{\infty} B_n(y)\,\frac{z^n}{n!} \qquad (B_0(y) = 1,\ B_1(y) = y - \tfrac{1}{2}, \ldots). \tag{1.3}$$

Bernoulli numbers: $B_n \equiv B_n(0)$ or $\dfrac{z}{\mathrm{e}^z - 1} \equiv \displaystyle\sum_{n=0}^{\infty} B_n\,\frac{z^n}{n!}$

$$(B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6}, \ldots;\ B_{2m+1} = 0 \text{ for } m = 1, 2, \ldots). \tag{1.4}$$

Euler numbers: $\dfrac{1}{\cosh z} \equiv \displaystyle\sum_{n=0}^{\infty} E_n\,\frac{z^n}{n!}$

$$(E_0 = 1,\ E_2 = -1,\ E_4 = 5, \ldots;\ E_{2m+1} = 0 \text{ for } m = 0, 1, \ldots). \tag{1.5}$$

Digamma function $\psi(x)$ and Euler's constant γ:

$$\psi(x) \stackrel{\text{def}}{=} [\Gamma'/\Gamma]\,(x); \qquad \gamma = -\psi(1) \approx 0.5772156649\,. \tag{1.6}$$