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(In-)Stability of Differential Inclusions

Notions, Equivalences, and Lyapunov-like Characterizations

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Preface

The fundamental theory that emerged from Aleksandr Mikhailovich Lyapunov's doctoral thesis [50] more than 100 years ago has been and still is the main tool to analyze stability properties of dynamical systems. Lyapunov or Lyapunov-like functions are monotone functions when evaluated along the solution of a dynamical system. Based on the monotonicity property, stability or instability of invariant sets can be concluded without the need to derive explicit solutions of the system dynamics.

In this monograph, existing results characterizing stability and stabilizability of the origin of differential inclusions through Lyapunov and control Lyapunov functions are reviewed and new characterizations for instability and destabilization characterized through Lyapunov-like arguments are derived. To distinguish between stability and instability, stability results are characterized through Lyapunov and control Lyapunov functions whereas instability is characterized through Chetaev and control Chetaev functions. In addition, similarities and differences between stability and instability and stabilizability and destabilizability of the origin of a differential inclusion are summarized. These connections are established by considering dynamics in forward time, in backward time, or by considering a scaled version of the differential inclusion. In total, the diagram shown in Fig. 1.1 is obtained, unifying new and existing results in a consistent notation.

As a last contribution of the monograph, ideas combining control Lyapunov and control Chetaev functions into a single framework are discussed. Through this approach, convergence (i.e., stability) and avoidance (i.e., instability) are guaranteed simultaneously.

The genesis of this monograph emerged from the preliminary results in [11], published as a conference paper in the proceedings of the 57th IEEE Conference on Decision and Control. Additionally, the ideas combining properties of control Lyapunov and control Chetaev functions rely on conference papers [13] and [14].

The authors would like to thank the anonymous reviewers for their helpful detailed comments and insights and for Example 2.17 and Remark 4.7 in particular.

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Chapter 1
Introduction

Abstract Lyapunov methods have been and still are one of the main tools to analyze stability properties of dynamical systems. In this monograph Lyapunov results characterizing stability and stabilizability of the origin of differential inclusions are reviewed. To characterize instability and destabilizability, Lyapunov-like functions, called Chetaev and control Chetaev functions in the monograph, are introduced. Based on their definition and by mirroring existing results on stability, analogue results for instability are derived. Moreover, by looking at the dynamics of a differential inclusion in backward time, similarities and differences between stability of the origin in forward time and instability in backward time, and vice versa, are discussed. Similarly, invariance of stability and instability properties of equilibria of differential equations with respect to a scaling are summarized. As a final result, ideas combining control Lyapunov and control Chetaev functions to simultaneously guarantee stability, i.e., convergence, and instability, i.e., avoidance, are outlined. The work is addressed at researchers working in control as well as graduate students in control engineering and applied mathematics.

Keywords Lyapunov methods · Differential inclusions · Stability of nonlinear systems · Instability of nonlinear systems · Stabilization/destabilization of nonlinear systems · Stabilizability and destabilizability

Lyapunov functions, originating from the work by Aleksandr Mikhailovich Lyapunov at the end of the 19th century [50], have been the main tool for the stability analysis of dynamical systems for more than a century. The strength of Lyapunov functions is that conclusions on stability properties of invariant sets of dynamical systems can be drawn without explicit knowledge of solutions and solely based on the time derivative of a Lyapunov function along solutions. While it is frequently nontrivial to find appropriate Lyapunov functions, in many cases it is significantly less challenging than the derivation of an explicit solution of the dynamical system.

Although Lyapunov functions were originally proposed by Lyapunov to provide sufficient conditions for stability properties of equilibria of differential equations, subsequent works by Barbashin and Krasovskii [8], Malkin [52], Massera [53], and

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Chetaev [19], for example, extended the results to other types of stability, applied them to more general dynamical systems, and began to address the converse question or the necessity of the various Lyapunov conditions. For an overview of contributions and the developments of Lyapunov methods we refer to the article [37].

A reference unifying many results on stability was written by Teel and Praly [69], showing that a closed set $\mathcal{A} \subset \mathbb{R}^n$ is $\mathcal{KL}$-stable with respect to two measures and with respect to the differential inclusion

$$\dot{x} \in F(x), \qquad x_0 \in \mathbb{R}^n, \tag{1.1}$$

with upper semicontinuous right-hand side $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, if and only if there exists a corresponding smooth Lyapunov function. In the case that $\mathcal{A}$ represents the origin and the measures are defined as the Euclidean norm, for example, then the results reduce to a characterization of asymptotic stability. Comparison functions introduced by Massera [53] and Hahn [35] (see also [36]) have become a modern tool in stability analysis and are used in $\mathcal{KL}$-stability to replace classical asymptotic stability definitions based on ε-δ formulations and convergence.

While the results in [69] focus on properties of *all solutions* of the differential inclusion and equivalent Lyapunov characterizations, results based on the *existence of at least one solution* with specific properties is analyzed through *control Lyapunov functions*. To avoid confusion, stability properties are usually called *strong* if *all solutions* are of interest and *weak* if the *existence of a solution* is of interest.

Control Lyapunov functions originate from the works of Artstein [5] and Sontag [63]. Sontag discusses the relationship between asymptotic controllability and the existence of a continuous but possibly nonsmooth positive definite function whose derivative along the solution decreases if the input of the control system is selected appropriately. Artstein expresses similar ideas for continuously differentiable functions and more restrictive dynamics.

While concepts like strong $\mathcal{KL}$-stability and asymptotic stability are equivalent to the existence of smooth Lyapunov functions the same connection cannot be established in the context of weak $\mathcal{KL}$-stability, asymptotic controllability or stabilizability and control Lyapunov functions. Famous examples include the Brockett integrator [16] and Artstein's circles [5], which show that asymptotic controllability does not necessarily imply the existence of a smooth control Lyapunov function. To overcome limitations due to assumptions on differentiability of candidate control Lyapunov functions, nonsmooth control Lyapunov functions have been introduced using the Dini derivative or proximal gradients (see [24], for example) instead of the directional derivative. Sontag and Sussman [66] showed that asymptotic controllability or stabilizability is equivalent to the existence of a continuous control Lyapunov function, and a control Lyapunov function was used by Clarke, Ledyaev, Sontag, and Subbotin [22] to derive a stabilizing controller based on this result. In [21], Clarke, Ledyaev, Rifford and Stern proved the existence of a continuous control Lyapunov function which is locally Lipschitz continuous on a domain excluding a neighborhood around the origin. Finally, weak $\mathcal{KL}$-stability of the origin of (1.1) was shown to be equivalent to the existence of a locally Lipschitz continuous control Lyapunov function by

Rifford in [59] and by Kellett and Teel in [38, 39]. In [59] it is additionally shown that there exists a semiconcave control Lyapunov function, a property which is stronger than Lipschitz continuity, while in [39] the results are not restricted to stability of the origin, but are applicable to more general invariant sets. For semiconcavity and properties of semiconcave control Lyapunov functions we refer to the references [10, 18, 20]. Here, differences between semiconcave functions and Lipschitz continuous functions will not be addressed and we focus on stability properties of the origin instead of more general sets.

While the terminology suggests that stability is a desirable phenomenon while instability is undesirable, in practical applications this is not necessarily true. For instance, in many practical applications modeled by dynamical systems or control systems, unsafe regions of the state space exist which should be avoided by the solutions of the system. Hence, it is desirable that such unsafe sets are unstable, which requires both the analysis of instability and the design of controllers that render such sets unstable.

This motivated the writing of this monograph, whose contributions are to mirror existing results on strong and weak $\mathcal{KL}$-stability to obtain corresponding instability results and to establish the diagram in Fig. 1.1.

In particular, instead of a Lyapunov function characterizing that all solutions of (1.1) converge to the origin, we give a Lyapunov-like function guaranteeing that all solutions go to infinity for $t \to \infty$. Similarly, results on the existence of one solution with certain properties analogous to control Lyapunov functions are derived. To acknowledge the contributions of Chetaev in the context of instability of differential equations and to differentiate between stability and instability properties, in this monograph Lyapunov-like functions characterizing instability properties are called Chetaev functions and control Chetaev functions, respectively. Precise definitions are provided in due course. Moreover, connections between stability properties in forward and backward time as well as invariance with respect to (positive) scaling of differential inclusions are discussed.

While results characterizing stability are essentially complete, instability and destabilization have not been studied to the same extent. We emphasize that while it may appear that such characterizations are easily obtained via time reversal, the situation is much more subtle. On the one hand, the fact that for stability the solutions usually flow towards a compact set, while for instability the solutions flow away from a compact set causes technical difficulties. On the other hand, as we will see in Sect. 4.3 and particularly in Corollary 4.15, weak (in)stability concepts are usually characterized by nonsmooth control Lyapunov or control Chetaev functions, respectively, and the corresponding generalized differential inequalities for nonsmooth functions cannot simply be reversed in time. In [26], so called anti-control Lyapunov functions, similar to control Chetaev functions discussed here, are introduced. However, necessary and sufficient conditions for the existence of anti-control Lyapunov functions are not derived. Instead, the anti-control Lyapunov functions are used to define destabilizing feedback laws based on Sontag's universal formula [64]. Additionally, in [27], control Chetaev functions have been proposed to characterize instability properties of control systems. However, the definition of these functions