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Hybrid High-Order Methods

A Primer with Applications to Solid Mechanics

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Preface

Hybrid High-Order (HHO) methods attach discrete unknowns to the cells and to the faces of the mesh. At the heart of their devising lie two intuitive ideas: (i) a local operator reconstructing in every mesh cell a gradient (and possibly a potential for the gradient) from the local cell and face unknowns and (ii) a local stabilization operator weakly enforcing in every mesh cell the matching of the trace of the cell unknowns with the face unknowns. These two local operators are then combined into a local discrete bilinear form, and the global problem is assembled cellwise as in standard finite element methods. HHO methods offer many attractive features: support of polyhedral meshes, optimal convergence rates, local conservation principles, a dimension-independent formulation, and robustness in various regimes (e.g., no volume-locking in linear elasticity). Moreover, their computational efficiency hinges on the possibility of locally eliminating the cell unknowns by static condensation, leading to a global transmission problem coupling only the face unknowns.

HHO methods were introduced in [77, 79] for linear diffusion and quasi-incompressible linear elasticity. A high-order method in mixed form sharing the same devising principles was introduced in [78], and shown in [6] to lead after hybridization to a HHO method with a slightly different, yet equivalent, writing of the stabilization. The realm of applications of HHO methods has been substantially expanded over the last few years. Developments in solid mechanics include nonlinear elasticity [26], hyperelasticity [1], plasticity [2, 3], poroelasticity [16, 27], Kirchhoff–Love plates [19], the Signorini [44], obstacle [59] and two-membrane contact [69] problems, Tresca friction [53], and acoustic and elastic wave propagation [33, 34]. Those related to fluid mechanics include convection-diffusion in various regimes [74], Stokes [6, 81], Navier–Stokes [23, 45, 82], Bingham [43], creeping non-Newtonian [24], and Brinkman [22] flows, flows in fractured porous media [47, 106], single-phase miscible flows [7], and elliptic [35] and Stokes [32] interface problems. Other interesting applications include the Cahn–Hilliard problem [49], Leray–Lions equations [72], elliptic multiscale problems [60], H^{-1} loads [95], spectral problems [38, 41], domains with curved boundary [21, 35, 36], and magnetostatics [48].

Bridges and unifying viewpoints emerged progressively between HHO methods and several other discretization methods which also attach unknowns to the mesh cells and faces. Already in the seminal work [79], a connection was established between the lowest-order HHO method and the hybrid finite volume method from [97] (and, thus, to the broader setting of hybrid mimetic mixed methods in [85]). Perhaps the most salient connection was made in [62] where HHO methods were embedded into the broad setting of Hybridizable Discontinuous Galerkin (HDG) methods [64]. One originality of equal-order HHO methods is the use of the (potential) reconstruction operator in the stabilization. Moreover, the analyses of HHO and HDG methods follow somewhat different paths, since the former relies on orthogonal projections, whereas the latter often invokes a more specific approximation operator [65]. We believe that the links between HHO and HDG methods are mutually beneficial, as, for instance, recent HHO developments can be transposed to the HDG setting. Weak Galerkin (WG) methods [148, 149], which were embedded into the HDG setting in [61 Sect. 6.6], are, thus, also closely related to HHO. WG and HHO were developed independently and share a common devising viewpoint combining reconstruction (called weak gradient in WG) and stabilization. Yet, the WG stabilization often relies on plain least-squares penalties, whereas the more sophisticated HHO stabilization is key to a higher-order consistency property. Furthermore, the work [62] also bridged HHO methods to the nonconforming virtual element method [10, 119]. Finally, the connection to the multiscale hybrid mixed method from [105] was uncovered in [46].

A detailed monograph on HHO methods appeared this year [73]. The present text is shorter and does not cover as many aspects of the analysis and applications of HHO methods. Its originality lies in targetting the material to computational mechanics without sacrificing mathematical rigor, while including on the one hand some mathematical results with their own specific twist and on the other hand numerical illustrations drawn from industrial examples. Moreover, several topics not covered in [73] are treated here: domains with curved boundary, hyperelasticity, plasticity, contact, friction, and wave propagation. The present material is organized into eight chapters: the first three gently introduce the basic principles of HHO methods on a linear diffusion problem, the following four present various challenging applications to solid mechanics, and the last one reviews implementation aspects.

This book is primarily intended for graduate students, researchers (in applied mathematics, numerical analysis, and computational mechanics), and engineers working in related fields of application. Basic knowledge of the devising and analysis of finite element methods is assumed. Special effort was made to streamline the presentation so as to pinpoint the essential ideas, address key mathematical aspects, present examples, and provide bibliographic pointers. This book can also be used as a support for lectures. As a matter of fact, its idea originated from a series of lectures given by one of the authors during the Workshop on Computational Modeling and Numerical Analysis (Petrópolis, Brasil, 2019).

We are thankful to many colleagues for stimulating discussions at various occasions. Special thanks go to G. Delay (Sorbonne University) and S. Lemaire (INRIA) for their careful reading of parts of this manuscript.

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Contents

Chapter 1
Getting Started: Linear Diffusion

The objective of this chapter is to gently introduce the hybrid high-order (HHO) method on one of the simplest model problems: the Poisson problem with homogeneous Dirichlet boundary conditions. Our goal is to present the key ideas underlying the devising of the method and state its main properties (most of them without proof). The keywords of this chapter are cell and face unknowns, local reconstruction and stabilization operators, elementwise assembly, static condensation, energy minimization, and equilibrated fluxes.

1.1 Model Problem

Let Ω be an open, bounded, connected, Lipschitz subset of $\mathbb{R}^d$ in space dimension $d \geq 2$. The one-dimensional case $d = 1$ can also be covered, and we refer the reader to Sect. 1.6 for an outline of HHO methods in this setting. Vectors in $\mathbb{R}^d$ and vector-valued functions are denoted in bold font, $\boldsymbol{a}{\cdot}\boldsymbol{b}$ denotes the Euclidean inner product between two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ and $\|{\cdot}\|_{\ell^2}$ the Euclidean norm in $\mathbb{R}^d$. Moreover, $\#S$ denotes the cardinality of a finite set S.

We use standard notation for the Lebesgue and Sobolev spaces; see, e.g., [30, Chaps. 4 and 8], [92, Chaps. 1–4], and [5, 96]. In particular, $L^2(\Omega)$ is the Lebesgue space composed of square-integrable functions over Ω, and $H^1(\Omega)$ is the Sobolev space composed of those functions in $L^2(\Omega)$ whose (weak) partial derivatives are square-integrable functions over Ω. Moreover, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ composed of functions with zero trace on the boundary $\partial\Omega$. Inner products and norms in these spaces are denoted by $(\cdot, \cdot)_{L^2(\Omega)}$, $\|{\cdot}\|_{L^2(\Omega)}$, $(\cdot, \cdot)_{H^1(\Omega)}$, and $\|{\cdot}\|_{H^1(\Omega)}$. Recall that for a real-valued function v:

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$$\|v\|^2_{L^2(\Omega)} := \int_\Omega v^2 \mathrm{d}x, \qquad \|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + \ell^2_\Omega \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{1.1}$$

where the length scale $\ell_\Omega := \operatorname{diam}(\Omega)$ (the diameter of Ω) is introduced to be dimensionally consistent. Owing to the Poincaré–Steklov inequality (a.k.a. Poincaré inequality; see [92, Remark 3.32] for a discussion on the terminology), there is $C_{\mathrm{PS}} > 0$ such that $C_{\mathrm{PS}}\|v\|_{L^2(\Omega)} \le \ell_\Omega \|\nabla v\|_{\boldsymbol{L}^2(\Omega)}$ for all $v \in H^1_0(\Omega)$.

The model problem we want to approximate in this chapter is the Poisson problem with source term $f \in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions, i.e., $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. The weak formulation of this problem reads as follows: Seek $u \in V := H^1_0(\Omega)$ such that

$$a(u, w) = \ell(w), \quad \forall w \in V, \tag{1.2}$$

with the following bounded bilinear and linear forms:

$$a(v, w) := (\nabla v, \nabla w)_{\boldsymbol{L}^2(\Omega)}, \qquad \ell(w) := (f, w)_{L^2(\Omega)}, \tag{1.3}$$

for all $v, w \in V$. Since we have $a(v, v) = \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)}$, the Poincaré–Steklov inequality implies that the bilinear form a is coercive on V. Hence, the model problem (1.2) is well-posed owing to the Lax–Milgram lemma.

1.2 Discrete Setting

In this section, we present the setting to formulate the HHO discretization of the model problem (1.2).

1.2.1 The Mesh

For simplicity, we assume in what follows that the domain Ω is a polyhedron in $\mathbb{R}^d$, so that its boundary is composed of a finite union of portions of affine hyperplanes with mutually disjoint interiors. The case of domains with a curved boundary is discussed in Sect. 3.2.2.

Since Ω is a polyhedron, it can be covered exactly by a mesh $\mathcal{T}$ composed of a finite collection of (open) polyhedral mesh cells T, all mutually disjoint, i.e., we have $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}$. Notice that by definition of a polyhedron, the mesh cells have straight edges if $d = 2$ and planar faces if $d = 3$. For a generic mesh cell $T \in \mathcal{T}$, its boundary is denoted by ∂T, its unit outward normal by $\boldsymbol{n}_T$, and its diameter by h_T. The mesh size is defined as the largest cell diameter in the mesh and is denoted by $h_{\mathcal{T}}$, and more simply by h when there is no ambiguity. When establishing error estimates, one is interested in the process $h \to 0$ corresponding to a sequence of successively

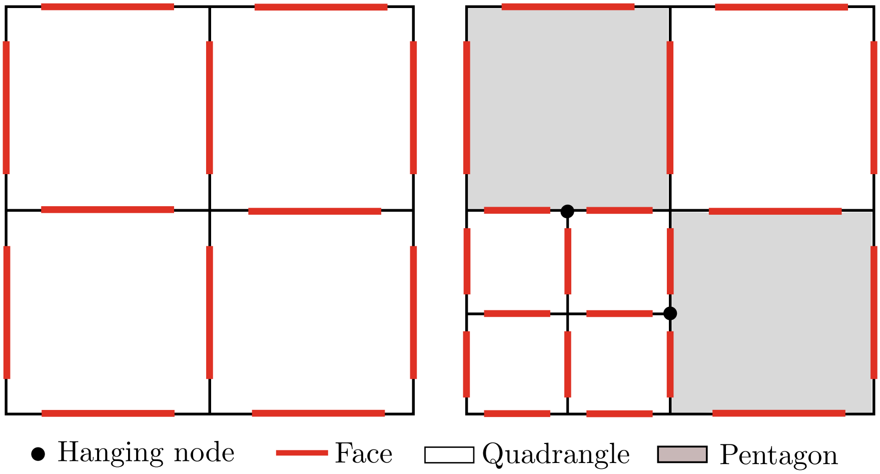

Fig. 1.1 Local refinement of a quadrilateral mesh; the mesh cells containing hanging nodes are treated as polygons (here, pentagons)

refined meshes. In this case, one needs to introduce a notion of shape-regularity for the mesh sequence. This notion is detailed in Sect. 2.1.

The possibility of handling meshes composed of polyhedral mesh cells is an attractive feature of HHO methods. For instance, it allows one to treat quite naturally the presence of hanging nodes arising from local mesh refinement; see Fig. 1.1 for an illustration. However, the reader can assume for simplicity that the mesh is composed of cells with a single shape, such as simplices (triangles in 2D, tetrahedra in 3D) or (rectangular) cuboids, without loosing anything essential in the understanding of the devising and analysis of HHO methods.

Besides the mesh cells, the mesh faces also play an important role in HHO methods. We say that the $(d-1)$-dimensional subset $F \subset \overline{\Omega}$ is a mesh face if F is a subset of an affine hyperplane, say H_F, such that the following holds: (i) either there are two distinct mesh cells $T_-, T_+ \in \mathcal{T}$ such that

$$F = \partial T_- \cap \partial T_+ \cap H_F, \tag{1.4}$$

and F is called a (mesh) interface; (ii) or there is one mesh cell $T_- \in \mathcal{T}$ such that

$$F = \partial T_- \cap \partial \Omega \cap H_F, \tag{1.5}$$

and F is called a (mesh) boundary face. The interfaces are collected in the set $\mathcal{F}^\circ$, the boundary faces in the set $\mathcal{F}^\partial$, so that the set

$$\mathcal{F} := \mathcal{F}^\circ \cup \mathcal{F}^\partial \tag{1.6}$$

collects all the mesh faces. For a mesh cell $T \in \mathcal{T}$, $\mathcal{F}_T$ denotes the collection of the mesh faces composing its boundary ∂T. Notice that the above definition of the mesh faces implies that each mesh face is straight in 2D and planar in 3D. Hence, for every mesh cell $T \in \mathcal{T}$, $\boldsymbol{n}_{T|F}$ is a constant vector on every face $F \in \mathcal{F}_T$. Notice also that the definitions (1.4) and (1.5) do not allow for the case of several coplanar faces that could be shared by two cells or a cell and the boundary, respectively; this choice is only made for simplicity.

1.2.2 Discrete Unknowns

The discrete unknowns in HHO methods are polynomials attached to the mesh cells and to the mesh faces. The idea is that the cell polynomials approximate the exact solution in the mesh cells, and that the face polynomials approximate the trace of the exact solution on the mesh faces (although they are not the trace of the cell polynomials). To ease the exposition, we consider here the equal-order HHO method where the cell and face polynomials have the same degree. Variants are considered in Sect. 3.2.1.

Let $k \geq 0$ be the polynomial degree. Let $\mathbb{P}_d^k$ be the space composed of d-variate (real-valued) polynomials of total degree at most k. For every mesh cell $T \in \mathcal{T}$, $\mathbb{P}_d^k(T)$ denotes the space composed of the restriction to T of the polynomials in $\mathbb{P}_d^k$. To define the $(d-1)$-variate polynomial space attached to a mesh face $F \in \mathcal{F}$ (which is a subset of $\mathbb{R}^d$), we consider an affine geometric mapping $\boldsymbol{T}_F : \mathbb{R}^{d-1} \to H_F$ (recall that H_F is the affine hyperplane in $\mathbb{R}^d$ supporting F). Then we set

$$\mathbb{P}_{d-1}^k(F) := \mathbb{P}_{d-1}^k \circ (\boldsymbol{T}_F^{-1})_{|F}. \tag{1.7}$$

It is easy to see that the definition of $\mathbb{P}_{d-1}^k(F)$ is independent of the choice of the affine geometric mapping $\boldsymbol{T}_F$. (Notice that defining polynomials on the mesh faces is meaningful since we are assuming $d \geq 2$.)

Let us first consider a local viewpoint. For every mesh cell $T \in \mathcal{T}$, we set

$$\hat{V}_T^k := \mathbb{P}_d^k(T) \times \mathbb{P}_{d-1}^k(\mathcal{F}_T), \qquad \mathbb{P}_{d-1}^k(\mathcal{F}_T) := \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F). \tag{1.8}$$

A generic element in $\hat{V}_T^k$ is denoted by $\hat{v}_T := (v_T, v_{\partial T})$. We shall systematically employ the hat notation to indicate a pair of (piecewise) functions, one attached to the mesh cell(s) and one to the mesh face(s). Notice that the trace of v_T on ∂T differs from $v_{\partial T}$; in particular, the former is a smooth function over ∂T, whereas the latter generally exhibits jumps from one face in $\mathcal{F}_T$ to an adjacent one. To define the global discrete HHO unknowns, we follow a similar paradigm; see Fig. 1.2.

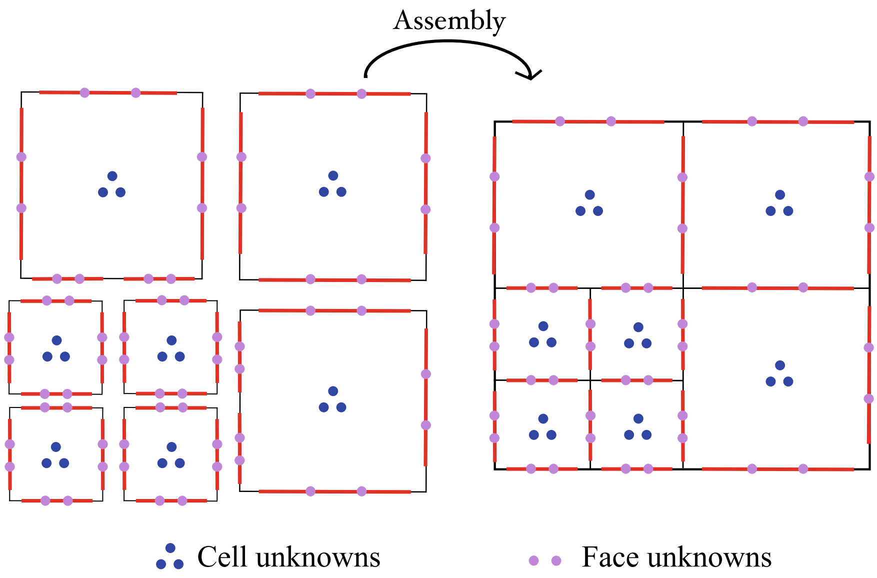

Fig. 1.2 Local (left) and global (right) unknowns for the HHO method ($d = 2, k = 1$). Each bullet on the faces and in the cells conventionally represents one basis function

Definition 1.1 (*HHO space*) The equal-order HHO space is defined as follows:

$$\hat{V}_h^k := V_{\mathcal{T}}^k \times V_{\mathcal{F}}^k, \qquad V_{\mathcal{T}}^k := \bigtimes_{T\in\mathcal{T}} \mathbb{P}_d^k(T), \qquad V_{\mathcal{F}}^k := \bigtimes_{F\in\mathcal{F}} \mathbb{P}_{d-1}^k(F). \tag{1.9}$$

We have $\dim(\hat{V}_h^k) = \binom{k+d}{d}\#\mathcal{T} + \binom{k+d-1}{d-1}\#\mathcal{F}$. □

A generic element in $\hat{V}_h^k$ is denoted by $\hat{v}_h := (v_{\mathcal{T}}, v_{\mathcal{F}})$ with $v_{\mathcal{T}} := (v_T)_{T\in\mathcal{T}}$ and $v_{\mathcal{F}} := (v_F)_{F\in\mathcal{F}}$. Notice that in general $v_{\mathcal{T}}$ is only piecewise smooth, i.e., it can jump across the mesh interfaces, and similarly $v_{\mathcal{F}}$ can jump from one mesh face to an adjacent one. Moreover, for all $\hat{v}_h \in \hat{V}_h^k$ and all $T \in \mathcal{T}$, it is convenient to localize the components of $\hat{v}_h$ associated with T and its faces by using the notation

$$\hat{v}_T := \big(v_T, v_{\partial T} := (v_F)_{F\in\mathcal{F}_T}\big) \in \hat{V}_T^k. \tag{1.10}$$

At this stage, a natural question that arises is how to reduce a generic function $v \in H^1(\Omega)$ (think of the weak solution to (1.2)) to some member of the discrete space $\hat{V}_h^k$. In the context of finite elements, this task is usually realized by means of the interpolation operator associated with the finite element. In the context of HHO methods, this task is realized in a simple way by considering L^2-orthogonal projections. Let $T \in \mathcal{T}$. Let $\Pi_T^k : L^2(T) \to \mathbb{P}_d^k(T)$ and $\Pi_{\partial T}^k : L^2(\partial T) \to \mathbb{P}_{d-1}^k(\mathcal{F}_T)$ be the L^2-orthogonal projections defined such that for all $v \in L^2(T)$ and all $w \in L^2(\partial T)$,