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Arno van den Essen
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Polynomial Automorphisms and the Jacobian Conjecture

New Results from the Beginning of the 21st Century

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For Toshio Kuroda and Michiko Kuroda
For Jennifer, Joseph, and Renee
For Sandra and Raïssa

Preface

In March 2015, I received an email from Clemens Heine, Executive Editor for Birkhäuser, asking me if I was interested in preparing an updated new edition of my book *Polynomial Automorphisms and the Jacobian Conjecture*, which appeared in 2000. Having thought some minutes about this proposal, I realized that many new exciting results have been obtained since the publication of that book some twenty years ago. To mention a few, the solution of Nagata's conjecture by Shestakov–Umirbaev, the complete solution of Hilbert's fourteenth problem by Kuroda, the equivalence of the Jacobian Conjecture and the Dixmier Conjecture by Tsuchimoto and independently by Belov-Kanel and Kontsevich, the symmetric reduction by de Bondt and myself, the theory of Mathieu–Zhao spaces by Wenhua Zhao, and finally the counterexamples to the Cancellation Problem in positive characteristic by Neena Gupta.

In order to give a good account of all these new developments, I asked the help of two experts, both excellent expository writers, Anthony J. Crachiola and Shigeru Kuroda. I asked Tony to cover the new results related to the Cancellation Problem and Shigeru to expose his results on Hilbert's fourteenth problem and the Shestakov–Umirbaev theory. During the writing of my own contributions, it became clear that together we would have enough material to write a new book.

The contents of this book are arranged as follows. In the first chapter, written by Shigeru Kuroda, the author extends the results obtained by Shestakov and Umirbaev and gives a useful criterion to decide if a given polynomial automorphism in dimension three is tame. His exposition is completely self-contained. As an application, it is shown that Nagata's famous automorphism is indeed wild, as originally conjectured by Nagata in 1972.

The second chapter, which is also written by Kuroda, gives a complete solution of Hilbert's fourteenth problem due to the author and includes his latest results.

The third chapter, written by Anthony J. Crachiola, discusses methods used to study problems related to polynomials in positive characteristic, including the Makar-Limanov and Derksen invariants, higher derivations, gradings, etcetera. These methods are used to explain the counterexamples to the Cancellation Problem in positive characteristic due to Neena Gupta.

The last two chapters form my own contribution. In Chap. 4, it is shown that the Jacobian Conjecture is equivalent to the Dixmier Conjecture, the Poisson Conjecture, and the Unimodular Conjecture. Furthermore, a p-adic formulation of the Jacobian Conjecture is given, due to Lipton and the author. At the end of the chapter, a false "proof" of the Jacobian Conjecture is constructed. It is left to the reader to find the error!

In Chap. 5, the theory of Mathieu–Zhao spaces, mainly due to Wenhua Zhao, is developed and various conjectures, all implying the Jacobian Conjecture, are discussed: the Vanishing Conjecture, the Generalized Vanishing Conjecture, the Image Conjecture, and the Gaussian Moment Conjecture.

After the last chapter, a list of corrections to my book [117] is given. At the time that book was published, it was the only one covering a large part of the young field of polynomial automorphisms, which belongs to the larger field of Affine Algebraic Geometry. During the last 20 years, several books in the field of Affine Algebraic Geometry were published. Two of them in the series *Encyclopaedia of Mathematical Sciences*, namely *Computational Invariant Theory* by Harm Derksen and Gregor Kemper [29] and *Algebraic Theory of Locally Nilpotent Derivations* by Gene Freudenburg [48]. A second enlarged edition of this monograph appeared in 2017. In 2016, the book *The Asymptotic Variety of Polynomial Maps* was published by Ronen Peretz [99], describing his approach to the two-dimensional Jacobian Conjecture.

Also, several survey papers covering a more geometric approach to polynomial maps appeared on arXiv, such as Masayoshi Miyanishi's *Lectures on Geometry and Topology of Polynomials - Surrounding the Jacobian Conjecture* [87] and the recent paper *On the Geometry of the Automorphism Groups of Affine Varieties* by Jean-Philippe Furter and Hanspeter Kraft [52]. Finally, we like to mention the very interesting recent paper *The Jacobian Conjecture Fails for Pseudo-Planes* by Adrien Dubouloz and Karol Palka [37].

Nijmegen, The Netherlands — Arno van den Essen
July 2019

Acknowledgments

It is my great pleasure to thank all those who have directly or indirectly supported my research so far. The works in Chap. 1 or 2 of this book might not have existed without them. Special thanks go to Arno van den Essen for inviting me to contribute to this book, published as a sequel of his book [117]. I was happy to accept his invitation, because [117] is special to me. About 20 years ago, I wrote my master's thesis about initial algebras based on my own interest [67]. At that time, I already had a feeling that research in that direction would yield important results. However, it was not easy to create mathematics as I was thinking. After groping in the dark, I arrived at Arno's book [117], where I found exactly what I had been looking for. In my view, [117] is a serious attempt to understand the real nature of polynomial rings, whose value will be timeless.

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Shigeru Kuroda

I am so grateful to Arno van den Essen for inviting me to be a part of this project. I especially appreciate his attitude toward writing: no deadlines, use your best judgment, and have fun! It was a lot of fun. Thanks to my home institution, Saginaw Valley State University, for providing an excellent work environment and all the support necessary to write. Special thanks to Lenny Makar-Limanov for teaching me about locally nilpotent derivations and for mentoring me in the years since. Finally, thanks to my wife Jennifer and our children Joseph and Renee. You all are the most perfect inspiration, motivation, and distraction all at once.

Anthony J. Crachiola

First of all, I want to thank Anthony J. Crachiola and Shigeru Kuroda for their beautiful contributions to this book. Without them, this “second volume” of *Polynomial Automorphisms and the Jacobian Conjecture* would never be written.

Next, I want to thank Michiel de Bondt. His many ideas have had a great influence both on my understanding of the Jacobian Conjecture and on the content of the last two chapters of this book.

Then there is of course Wenhua Zhao. His beautiful ideas shaped the last chapter and enriched our view of the Jacobian Conjecture. Furthermore, I like to thank him for bringing my wife, Sandra, in contact with the Chinese acupuncturist Dr. Zhang during the conference we organized in Tianjin in July 2014. This meeting and several later meetings with Dr. Zhang changed our lives and that is what I am the most grateful for.

Also, I like to thank Hongbo Guo for inviting me to Tianjin several times. Next, I want to thank Pascal Adjamagbo and Harm Derksen for our long friendship and fruitful mathematical cooperation. I want to thank Jan Schoone for reading several parts of this book, giving suggestions for its improvement, and helping me with the editing of the final text.

A special word of thanks to Clemens Heine from Birkhäuser for the excellent cooperation during this book project over the years.

Finally, I like to thank my wife, Sandra, and our daughter, Raissa: the joy and happiness they bring into my life every day cannot be expressed in words.

Arno van den Essen

Contents

The Shestakov–Umirbaev Theory and Nagata's Conjecture

1

1.1 The Shestakov–Umirbaev Theory

As discussed in the introduction of [117], Nagata [94] conjectured that there exist wild polynomial automorphisms in three variables (Conjecture 1.1.1). In 2004, after [117] was published, Shestakov–Umirbaev [109, 110] showed that the conjecture is true if the coefficient field is of characteristic zero. The purpose of this chapter is to give an introduction to the Shestakov–Umirbaev theory. We give a self-contained proof of a wildness criterion of polynomial automorphisms in three variables (Sect. 1.1.4). Using this criterion, we can easily check the wildness of Nagata's famous automorphism (Exercise 11).

We emphasize that the Shestakov–Umirbaev theory is not a theory of wild automorphisms, but a theory of tame automorphisms. This may sound strange. However, if one would like to say that an automorphism is wild, i.e., not tame, one first needs to know what the tame automorphisms are. Shestakov and Umirbaev obtained a condition which *every tame automorphism satisfies*. It is not difficult to check that Nagata's automorphism does not satisfy the condition.

We note that the "Shestakov–Umirbaev theory" we present here is the one modified by the author [73, 74]. The main differences from the original theory are as follows:

- One of the most important tools in the Shestakov–Umirbaev theory is a certain inequality for estimating degrees of polynomials (cf. [109], Theorem 3). The author improved this result in [73], which makes the argument more simple and precise.
- Shestakov and Umirbaev considered total degrees, while the author considered more generally weighted degrees. This generalization is of great help in applications. Also, the proof of Nagata's conjecture becomes slightly simpler by using an "independent" weight (cf. Sect. 1.1.2).

A. van den Essen et al., *Polynomial Automorphisms and the Jacobian Conjecture*, Frontiers in Mathematics, https://doi.org/10.1007/978-3-030-60535-3_1

Notation and Convention Throughout this chapter, let k be a field of characteristic zero except for Sect. 1.1.1, and let $k[\boldsymbol{x}] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. We often use the letters F and G to denote the elements of $k[\boldsymbol{x}]^r$ for $r \geq 1$, and write $F = (f_1, \ldots, f_r)$ and $G = (g_1, \ldots, g_r)$ without mentioning it. We write $S_l := \{f_1, \ldots, \widehat{f_l}, \ldots, f_r\}$ for $1 \leq l \leq r$. For each permutation $\sigma \in \mathfrak{S}_r$, we define $F_\sigma := (f_{\sigma(1)}, \ldots, f_{\sigma(r)})$. We denote by $\mathscr{T}$ the set of $F \in k[\boldsymbol{x}]^3$ such that f_1, f_2, and f_3 are algebraically independent over k. We identify each $F \in k[\boldsymbol{x}]^r$ with the substitution map $k[x_1, \ldots, x_r] \to k[\boldsymbol{x}]$ defined by $x_i \mapsto f_i$ for each i. Then, for $F \in k[\boldsymbol{x}]^n$ and $G \in k[\boldsymbol{x}]^r$, the composite

$$k[x_1, \ldots, x_r] \xrightarrow{G} k[\boldsymbol{x}] \xrightarrow{F} k[\boldsymbol{x}]$$

is written as $FG = (g_1(f_1, \ldots, f_n), \ldots, g_r(f_1, \ldots, f_n))$.

1.1.1 Nagata's Conjecture

In this section, let k be any field. Denote by $\mathrm{Aut}_k\, k[\boldsymbol{x}]$ the automorphism group of the k-algebra $k[\boldsymbol{x}]$. We remark that $F \in k[\boldsymbol{x}]^n$ belongs to $\mathrm{Aut}_k\, k[\boldsymbol{x}]$ if and only if $k[f_1, \ldots, f_n] = k[\boldsymbol{x}]$, i.e., $F : k[\boldsymbol{x}] \to k[\boldsymbol{x}]$ is surjective, since $\mathrm{tr.deg}_k(k[\boldsymbol{x}]/\ker F) = n$ implies $\ker F = (0)$. For example, we have

$$(x_1, \ldots, x_n)A + (b_1, \ldots, b_n) \in \mathrm{Aut}_k\, k[\boldsymbol{x}] \tag{1.1.1}$$

for each $A \in GL(n, k)$ and $b_1, \ldots, b_n \in k$, and

$$(x_1, \ldots, x_{l-1}, x_l + f, x_{l+1}, \ldots, x_n) \in \mathrm{Aut}_k\, k[\boldsymbol{x}] \tag{1.1.2}$$

for each $1 \leq l \leq n$ and $f \in k[x_1, \ldots, \widehat{x_l}, \ldots, x_n]$. Automorphisms of $k[\boldsymbol{x}]$ as in (1.1.1) and (1.1.2) are called **affine automorphisms** and **elementary automorphisms**, respectively. The **tame subgroup** $T(n, k)$ is the subgroup of $\mathrm{Aut}_k\, k[\boldsymbol{x}]$ generated by all the affine automorphisms and elementary automorphisms. We say that $F \in \mathrm{Aut}_k\, k[\boldsymbol{x}]$ is **tame** if F belongs to $T(n, k)$, and **wild** otherwise.

Clearly, we have $\mathrm{Aut}_k\, k[x_1] = T(1, k)$. Due to Jung [63] and van der Kulk [129], $\mathrm{Aut}_k\, k[x_1, x_2] = T(2, k)$ also holds true. Nagata conjectured that $\mathrm{Aut}_k\, k[x_1, x_2, x_3] \neq T(3, k)$ and gave the following candidate example of a wild automorphism (see [94], Part 2, Conjecture 3.1).

Conjecture 1.1.1 (Nagata) $(f_1, f_2, f_3) \not\in T(3, k)$, where

$$\begin{aligned} f_1 &= x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3 \\ f_2 &= x_2 + (x_1x_3 + x_2^2)x_3 \\ f_3 &= x_3. \end{aligned} \tag{1.1.3}$$

Exercise 1 Show that $k[f_1, f_2, f_3] = k[x_1, x_2, x_3]$ for f_1, f_2, and f_3 in (1.1.3). [Note that $x_1x_3 + x_2^2 = f_1f_3 + f_2^2 \in k[f_1, f_2, f_3]$.]

About 30 years after Nagata's book [94] was published, Shestakov–Umirbaev [109, 110] finally settled this conjecture when k is of characteristic zero.

Theorem 1.1.2 (Shestakov–Umirbaev) *Conjecture 1.1.1 is true if k is of characteristic zero.*

At present, however, it is not known whether $\mathrm{Aut}_k\, k[\boldsymbol{x}] = T(n, k)$ holds when $n = 3$ and k is of positive characteristic, or when $n \geq 4$.

Exercise 2

(1) Show that $T(n, k)$ is generated by the automorphisms of the form

$$(x_1, \dots, x_{l-1}, \alpha x_l + f, x_{l+1}, \dots, x_n),$$

where $1 \leq l \leq n$, $\alpha \in k^*$ and $f \in k[x_1, \dots, \widehat{x_l}, \dots, x_n]$.
[By linear algebra, every element of $GL(n, k)$ is changed to the identity matrix by an iteration of column operations.]

(2) Let $E(n, k)$ be the subgroup of $\mathrm{Aut}_k\, k[\boldsymbol{x}]$ generated by all the elementary automorphisms, and let

$$D(n, k) := \{(\alpha_1x_1, \dots, \alpha_nx_n) \mid \alpha_1, \dots, \alpha_n \in k^*\}.$$

Show that $T(n, k) = D(n, k)E(n, k)$.

1.1.2 Weighted Grading

Let Γ be a **totally ordered $\mathbb{Q}$-vector space**, i.e., a $\mathbb{Q}$-vector space equipped with a total ordering such that

$$\alpha \leq \beta \text{ implies } \alpha + \gamma \leq \beta + \gamma \text{ for each } \alpha, \beta, \gamma \in \Gamma. \tag{1.1.4}$$

For example, $\mathbb{R}$ is a $\mathbb{Q}$-vector space and satisfies (1.1.4) for the standard ordering. The $\mathbb{Q}$-vector space $\mathbb{Q}^n$ satisfies (1.1.4) for the lexicographic order.

We fix a **weight** $\mathbf{w} = (w_1, \ldots, w_n) \in \Gamma^n$ with $w_1, \ldots, w_n > 0$ and consider the **w-weighted grading** $k[\boldsymbol{x}] = \bigoplus_{\gamma\in\Gamma} k[\boldsymbol{x}]_\gamma$, where $k[\boldsymbol{x}]_\gamma$ is the k-vector space generated by monomials $x_1^{i_1} \cdots x_n^{i_n} \in k[\boldsymbol{x}]$ with $i_1w_1+\cdots+i_nw_n = \gamma$. Note that $k[\boldsymbol{x}]_\alpha k[\boldsymbol{x}]_\beta \subset k[\boldsymbol{x}]_{\alpha+\beta}$ for each $\alpha, \beta \in \Gamma$.

Exercise 3 Assume that $h \in k[\boldsymbol{x}]_\delta$, $h_1 \in k[\boldsymbol{x}]_{\delta_1}, \ldots, h_r \in k[\boldsymbol{x}]_{\delta_r}$ are nonzero, where $\delta, \delta_1, \ldots, \delta_r \in \Gamma$. Show that $h \in k[h_1, \ldots, h_r]$ implies

$$\delta \in \mathbb{Z}_{\geq 0}\delta_1 + \cdots + \mathbb{Z}_{\geq 0}\delta_r := \{a_1\delta_1 + \cdots + a_r\delta_r \mid a_1, \ldots, a_r \in \mathbb{Z}_{\geq 0}\}.$$

Now, take any $f = \sum_{\gamma\in\Gamma} f_\gamma \in k[\boldsymbol{x}]$, where $f_\gamma \in k[\boldsymbol{x}]_\gamma$. If $f \neq 0$, we define the **w-degree** of f by

$$\deg_{\mathbf{w}} f := \max\{\gamma \in \Gamma \mid f_\gamma \neq 0\}$$

and set $f^{\mathbf{w}} := f_{\deg_{\mathbf{w}} f}$. If $f = 0$, we define $\deg f := -\infty$ and $f^{\mathbf{w}} := 0$. Then, for each $f, g \in k[\boldsymbol{x}]$, we have

$$\deg_{\mathbf{w}} fg = \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} g \qquad \text{and} \qquad (fg)^{\mathbf{w}} = f^{\mathbf{w}} g^{\mathbf{w}}. \tag{1.1.5}$$

Example 1.1.3

(1) If $\mathbf{w} = (1, \ldots, 1) \in \mathbb{R}^n$, then $\deg_{\mathbf{w}} f$ is the same as the total degree of f.
(2) Let $\Gamma = \mathbb{Q}^3$ with the lexicographic order, and let $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the coordinate unit vectors of $\mathbb{Q}^3$. Then, for f_1, f_2, and f_3 in (1.1.3), we have $f_1^{\mathbf{w}} = -x_1^2x_3^3$, $f_2^{\mathbf{w}} = x_1x_3^2$, and $f_3^{\mathbf{w}} = x_3$, and

$$\deg_{\mathbf{w}} f_1 = (2, 0, 3), \quad \deg_{\mathbf{w}} f_2 = (1, 0, 2), \quad \text{and} \quad \deg_{\mathbf{w}} f_3 = (0, 0, 1).$$

Exercise 4 Show the following:

(1) $L := \{f/g \mid f, g \in k[\boldsymbol{x}]_\gamma,\ g \neq 0,\ \gamma \in \Gamma\}$ is a subfield of the rational function field $k(x_1, \ldots, x_n)$.
(2) Every element h of $k[\boldsymbol{x}] \setminus k$ is transcendental over L. [For any $\gamma \in \Gamma$, $l \geq 0$ and $c_0, \ldots, c_l \in k[\boldsymbol{x}]_\gamma$ with $c_l \neq 0$, we have $(\sum_{i=0}^{l} c_i h^i)^{\mathbf{w}} = c_l^{\mathbf{w}} (h^{\mathbf{w}})^l \neq 0$.]

We say that $\mathbf{w}$ is **independent** if $w_1, \ldots, w_n$ are linearly independent over $\mathbb{Q}$. For example, $\mathbf{w} = (1, \sqrt{2}, \sqrt{3}) \in \mathbb{R}^3$ is independent. If $\mathbf{w}$ is independent, then $\deg_{\mathbf{w}} x_1^{i_1} \cdots x_n^{i_n} = \sum_{l=1}^{n} i_l w_l$'s are different for different $(i_1, \ldots, i_n)$'s. Hence, $f^{\mathbf{w}}$ is always a monomial. Moreover, we have

$$\deg_{\mathbf{w}} f = \deg_{\mathbf{w}} g \iff f^{\mathbf{w}} \approx g^{\mathbf{w}} \qquad \text{for} \qquad f, g \in k[\boldsymbol{x}]. \tag{1.1.6}$$

Here, we write $f \approx g$ (resp., $f \not\approx g$) if f and g are linearly dependent (resp., linearly independent) over k.

Exercise 5 Consider the following conditions for $f_1, \ldots, f_r \in k[\boldsymbol{x}] \setminus \{0\}$:

(a) $\deg_{\mathbf{w}} f_1, \ldots, \deg_{\mathbf{w}} f_r$ are linearly independent over $\mathbb{Q}$.
(b) $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ are algebraically independent over k.

(1) Show that (a) implies (b).
[$\deg_{\mathbf{w}}(f_1^{\mathbf{w}})^{i_1} \cdots (f_r^{\mathbf{w}})^{i_r}$'s are different for different $(i_1, \ldots, i_r)$'s.]
(2) Show that (b) implies (a) when $\mathbf{w}$ is independent.
[If (a) is false, then $\deg_{\mathbf{w}} f_1^{i_1} \cdots f_r^{i_r} = \deg_{\mathbf{w}} f_1^{j_1} \cdots f_r^{j_r}$ for some distinct $(i_1, \ldots, i_r), (j_1, \ldots, j_r) \in (\mathbb{Z}_{\geq 0})^r$. This implies $(f_1^{i_1} \cdots f_r^{i_r})^{\mathbf{w}} \approx (f_1^{j_1} \cdots f_r^{j_r})^{\mathbf{w}}$ by (1.1.6).]

Proposition 1.1.4 *Let $f \in k[\boldsymbol{x}]_\alpha \setminus k$ and $g \in k[\boldsymbol{x}]_\beta \setminus k$, where $\alpha, \beta \in \Gamma$. If f and g are algebraically dependent over k, then $f^q \approx g^p$ holds for some $p, q \geq 1$ with* $\gcd(p, q) = 1$.

Proof Note that $f^{\mathbf{w}} = f$ and $g^{\mathbf{w}} = g$. Hence, by Exercise 5 (1), we can find $p, q \geq 1$ such that $q \deg f = p \deg g$ and $\gcd(p, q) = 1$. Then, f^q/g^p lies in the field L of Exercise 4. Hence, g is transcendental over $k(f^q/g^p)$. Since $\operatorname{tr.deg}_k k(f^q/g^p, g) = 1$, it follows that $f^q/g^p \in k$. $\square$

We study tuples of elements of $k[\boldsymbol{x}]$ by means of $\mathbf{w}$-weighted gradings. For each $F \in k[\boldsymbol{x}]^r$ with $r \geq 1$, we define

$$\deg_{\mathbf{w}} F := \deg_{\mathbf{w}} f_1 + \cdots + \deg_{\mathbf{w}} f_r \quad \text{and} \quad F^{\mathbf{w}} := (f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}).$$

Now, let $F \in \operatorname{Aut}_k k[\boldsymbol{x}]$. Then, the Jacobian of F is nonzero. Hence, we have $\prod_{l=1}^n (\partial f_l / \partial x_{\sigma(l)}) \neq 0$ for some $\sigma \in \mathfrak{S}_n$. Then, f_l depends on $x_{\sigma(l)}$ for each l, and so $\deg_{\mathbf{w}} f_l \geq w_{\sigma(l)}$ for each l. Thus, we obtain

$$\deg_{\mathbf{w}} F \geq w_1 + \cdots + w_n =: |\mathbf{w}|. \tag{1.1.7}$$

Proposition 1.1.5 *If the equality holds in (1.1.7), then* $F, F^{\mathbf{w}} \in T(n, k)$.

Proof We only prove the case $n = 3$. The general case is left to the reader. Without loss of generality, we may assume that $\deg_{\mathbf{w}} f_i = w_i$ for each i, and $\mathbf{w}$ satisfies one of the following:

(1) $w_1 = w_2 = w_3$. (2) $w_1 < w_2 < w_3$. (3) $w_1 = w_2 < w_3$. (4) $w_3 < w_1 = w_2$.

Then, F and $F^{\mathbf{w}}$ are tame automorphisms of the following types, where $\mathrm{Aff}(2, k)$ denotes the set of affine automorphisms of $k[x_1, x_2]$:

(1) affine automorphism.
(2) (g_1, g_2, g_3), where $g_i \in k^* x_i + k[x_1, \ldots, x_{i-1}]$ for $i = 1, 2, 3$.
(3) (G', g), where $G' \in \mathrm{Aff}(2, k)$ and $g \in k^* x_3 + k[x_1, x_2]$.
(4) $(G' + G'', g)$, where $G' \in \mathrm{Aff}(2, k)$, $G'' \in k[x_3]^2$, and $g \in k^* x_3 + k$. □

1.1.3 Initial Algebras and Elementary Reductions

For each k-subalgebra A of $k[\boldsymbol{x}]$, we define $A^{\mathbf{w}}$ to be the k-vector space generated by $f^{\mathbf{w}}$ for $f \in A$. Then, $A^{\mathbf{w}}$ is a k-subalgebra of $k[\boldsymbol{x}]$ by (1.1.5), which we call the **initial algebra** of A.

For $h \in k[\boldsymbol{x}] \setminus \{0\}$, we have $h^{\mathbf{w}} \in A^{\mathbf{w}}$ if there exists $\phi \in A$ such that $h^{\mathbf{w}} = \phi^{\mathbf{w}}$, i.e., $\deg_{\mathbf{w}}(h - \phi) < \deg h$. The converse is also true.

Exercise 6 Show that $h^{\mathbf{w}} \in A^{\mathbf{w}}$ implies $h^{\mathbf{w}} = \phi^{\mathbf{w}}$ for some $\phi \in A$.

For each $f_1, \ldots, f_r \in k[\boldsymbol{x}]$, we have $k[f_1, \ldots, f_r]^{\mathbf{w}} \supset k[f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}]$. The equality holds if $r = 1$. In general, however, it is difficult to determine the generators of the k-algebra $k[f_1, \ldots, f_r]^{\mathbf{w}}$.

Exercise 7 (Robbiano–Sweedler [103]) Let $A = k[x_1 + x_2, x_1x_2, x_1x_2^2]$.

(1) Show that $x_1x_2^l \in A$ for all $l \geq 1$. [$x_1x_2^l = x_1x_2^{l-1}(x_1 + x_2) - x_1x_2^{l-2} \cdot x_1x_2$]
(2) Show that $A \cap k[x_2] = k$ and $A^{\mathbf{w}} = k + x_1k[x_1, x_2]$ for $\mathbf{w} = (1, 0)$.

Note: The k-algebra $A^{\mathbf{w}} = k + x_1k[x_1, x_2]$ is not finitely generated.

Now, write $\phi \in k[f_1, \ldots, f_r] \setminus \{0\}$ as $\phi = \sum_{i_1, \ldots, i_r} u_{i_1, \ldots, i_r} f_1^{i_1} \cdots f_r^{i_r}$, where $u_{i_1, \ldots, i_r} \in k$. Then, $\deg_{\mathbf{w}} \phi$ is at most

$$\delta := \max\{\deg_{\mathbf{w}} f_1^{i_1} \cdots f_r^{i_r} \mid u_{i_1, \ldots, i_r} \neq 0\},\ \text{the } \textbf{apparent w-degree} \text{ of } \phi.$$

We define $\phi' := \sum' u_{i_1,\ldots,i_r}(f_1^{\mathbf{w}})^{i_1}\cdots(f_r^{\mathbf{w}})^{i_r}$, where the sum $\sum'$ is taken over $i_1,\ldots,i_r \geq 0$ with $\deg_{\mathbf{w}} f_1^{i_1}\cdots f_r^{i_r} = \delta$.

Remark 1.1.6

(i) $\phi' \neq 0$ if and only if $\deg_{\mathbf{w}} \phi = \delta$.
(ii) If $\phi' \neq 0$, then $\phi^{\mathbf{w}} = \phi'$, so $\phi^{\mathbf{w}}$ belongs to $k[f_1^{\mathbf{w}},\ldots,f_r^{\mathbf{w}}]$.

If $f_1^{\mathbf{w}},\ldots,f_r^{\mathbf{w}}$ are algebraically independent over k, then ϕ' is always nonzero. Hence, the following lemma holds.

Lemma 1.1.7 *Let $f_1,\ldots,f_r \in k[\boldsymbol{x}]$ be such that $f_1^{\mathbf{w}},\ldots,f_r^{\mathbf{w}}$ are algebraically independent over k. Then, we have $k[f_1,\ldots,f_r]^{\mathbf{w}} = k[f_1^{\mathbf{w}},\ldots,f_r^{\mathbf{w}}]$.*

The following notion is important in studying the elements of $\mathrm{Aut}_k\, k[\boldsymbol{x}]$.

Definition 1.1.8 We say that $F \in (k[\boldsymbol{x}]\setminus\{0\})^r$ admits an **elementary reduction** if there exists $1 \leq l \leq r$ such that $f_l^{\mathbf{w}} \in k[f_1,\ldots,\widehat{f_l},\ldots,f_r]^{\mathbf{w}}$, i.e., $\deg_{\mathbf{w}}(f_l-\phi) < \deg_{\mathbf{w}} f_l$ for some $\phi \in k[f_1,\ldots,\widehat{f_l},\ldots,f_r]$ (cf. Exercise 6). We call

$$F' := (f_1,\ldots,f_{l-1}, f_l-\phi, f_{l+1},\ldots,f_r)$$

an **elementary reduction** of F.

Note that $\deg_{\mathbf{w}} F' < \deg_{\mathbf{w}} F$, and $F' = FE$ holds for

$$E := (x_1,\ldots,x_{l-1}, x_l-\psi, x_{l+1},\ldots,x_r),$$

where $\psi \in k[x_1,\ldots,\widehat{x_l},\ldots,x_r]$ is such that $\phi = F(\psi)$.

Remark 1.1.9 $F \in (k[\boldsymbol{x}]\setminus\{0\})^r$ admits an elementary reduction if and only if there exists an elementary automorphism E of $k[x_1,\ldots,x_r]$ such that $\deg_{\mathbf{w}} FE < \deg_{\mathbf{w}} F$. Hence, by (1.1.7), $F \in \mathrm{Aut}_k\, k[\boldsymbol{x}]$ admits no elementary reduction if $\deg_{\mathbf{w}} F = |\mathbf{w}|$.

Proposition 1.1.10 *If $F \in \mathrm{Aut}_k\, k[\boldsymbol{x}]$ satisfies the following conditions, then we have $\deg_{\mathbf{w}} F > |\mathbf{w}|$, and F admits no elementary reduction*:

(e1) *$f_1^{\mathbf{w}},\ldots,f_n^{\mathbf{w}}$ are algebraically dependent over k, but any $n-1$ of them are algebraically independent over k.*
(e2) *$f_i^{\mathbf{w}} \not\in k[f_1^{\mathbf{w}},\ldots,\widehat{f_i^{\mathbf{w}}},\ldots,f_n^{\mathbf{w}}]$ for $i = 1,\ldots,n$.*

Proof By (e1), we have $F^{\mathbf{w}} \not\in \mathrm{Aut}_k\, k[\boldsymbol{x}]$. This implies $\deg_{\mathbf{w}} F > |\mathbf{w}|$ by Proposition 1.1.5. By Lemma 1.1.7, the last part of (e1) implies

$$k[f_1, \ldots, \widehat{f_i}, \ldots, f_n]^{\mathbf{w}} = k[f_1^{\mathbf{w}}, \ldots, \widehat{f_i^{\mathbf{w}}}, \ldots, f_n^{\mathbf{w}}]$$

for $i = 1, \ldots, n$. Hence, F admits no elementary reduction by (e2). □

Corollary 1.1.11 *Assume that* $\mathbf{w}$ *is independent. If* $F \in \mathrm{Aut}_k\, k[\boldsymbol{x}]$ *satisfies the following conditions, then we have* $\deg_{\mathbf{w}} F > |\mathbf{w}|$*, and* F *admits no elementary reduction*:

(E1) $\deg_{\mathbf{w}} f_1, \ldots, \deg_{\mathbf{w}} f_n$ *are linearly dependent over* $\mathbb{Q}$*, but any* $n-1$ *of them are linearly independent over* $\mathbb{Q}$.

(E2) $\deg_{\mathbf{w}} f_i \not\in \sum_{j \neq i} \mathbb{Z}_{\geq 0} \deg_{\mathbf{w}} f_j$ *for* $i = 1, \ldots, n$.

Proof Since $\mathbf{w}$ is independent, (E1) is equivalent to (e1) by Exercise 5. By Exercise 3, (E2) implies (e2). □

Finally, we discuss well-orderedness property of $\mathbf{w}$-degrees. The following exercise is essential.

Exercise 8

(1) Show that every infinite sequence of elements of $\mathbb{Z}_{\geq 0}$ has an infinite, non-decreasing subsequence.
(2) Show that every infinite sequence $(a_i)_i$ of elements of $(\mathbb{Z}_{\geq 0})^n$ has an infinite subsequence $(a_{i_l})_l$ such that $a_{i_{l+1}} - a_{i_l} \in (\mathbb{Z}_{\geq 0})^n$ for all l.

Lemma 1.1.12 $\sum_{i=1}^{n} \mathbb{Z}_{\geq 0} w_i$ *is a well-ordered subset of* Γ.

Proof Suppose that the lemma is false. Then, $\sum_{i=1}^{n} \mathbb{Z}_{\geq 0} w_i$ contains an infinite, strictly decreasing sequence $\mathbf{a} = (a_{i,1}w_1 + \cdots + a_{i,n}w_n)_{i=1}^{\infty}$, where $a_{i,j} \in \mathbb{Z}_{\geq 0}$. Since $w_1, \ldots, w_n > 0$, we know by Exercise 8 (2) that $\mathbf{a}$ has an infinite, non-decreasing subsequence, which is absurd. □

Remark 1.1.13 Let A be a k-subalgebra of $k[\boldsymbol{x}]$, and $h \in k[\boldsymbol{x}] \setminus A$. Then, $\{\deg_{\mathbf{w}} f \mid f \in h + A\}$ has a least element by Lemma 1.1.12. Hence, there exists $f \in h + A$ such that $f^{\mathbf{w}} \not\in A^{\mathbf{w}}$ in view of Exercise 6.

Exercise 9 In the situation of Remark 1.1.13, let $g \in h + A$ be such that $\deg_{\mathbf{w}} g > \min\{\deg_{\mathbf{w}} f \mid f \in h + A\}$. Show that $g^{\mathbf{w}}$ belongs to $A^{\mathbf{w}}$. [Take $f \in h + A$ with $\deg_{\mathbf{w}} f < \deg_{\mathbf{w}} g$. Then, $g^{\mathbf{w}} = (g - f)^{\mathbf{w}}$ and $g - f \in A$.]

By Remark 1.1.13 and Exercise 9, the following holds for each $g \in h + A$:

$$\deg_{\mathbf{w}} g = \min\{\deg_{\mathbf{w}} f \mid f \in h + A\} \iff g^{\mathbf{w}} \notin A^{\mathbf{w}}. \tag{1.1.8}$$

1.1.4 Wildness Criterion

Assume that $n \geq 3$. Recall that $\mathscr{T}$ is the set of $F \in k[\boldsymbol{x}]^3$ such that f_1, f_2, and f_3 are algebraically independent over k.

Definition 1.1.14 (Kuroda) We say that the pair $(F, G) \in \mathscr{T}^2$ satisfies the **Shestakov–Umirbaev condition** if the following conditions hold:

(SU1) $g_1 \in f_1 + kf_3^2 + kf_3$, $g_2 \in f_2 + kf_3$ and $g_3 \in f_3 + k[g_1, g_2]$.
(SU2) $\deg_{\mathbf{w}} f_1 \leq \deg_{\mathbf{w}} g_1$ and $\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} g_2$.
(SU3) $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ for some odd number $s \geq 3$.
(SU4) $\deg_{\mathbf{w}} f_3 \leq \deg_{\mathbf{w}} g_1$ and $f_3^{\mathbf{w}} \notin k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$.
(SU5) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} f_3$.
(SU6) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} g_1 - \deg_{\mathbf{w}} g_2 + \deg_{\mathbf{w}} dg_1 \wedge dg_2$.

Here, $\deg_{\mathbf{w}} dg_1 \wedge dg_2$ denotes the maximum among

$$\deg_{\mathbf{w}} \left| \frac{\partial(g_1, g_2)}{\partial(x_i, x_j)} \right| x_i x_j \quad \text{for} \quad 1 \leq i < j \leq n. \tag{1.1.9}$$

Exercise 10 In the situation of Definition 1.1.14, show the following:

(1) If $\deg_{\mathbf{w}} f_1 = \deg_{\mathbf{w}} g_1$, then $2 \deg_{\mathbf{w}} f_1 = s \deg_{\mathbf{w}} f_2$.
(2) If $\deg_{\mathbf{w}} f_1 < \deg_{\mathbf{w}} g_1$, then $s \deg_{\mathbf{w}} f_2 = 4 \deg_{\mathbf{w}} f_3$. [We have $g_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$ or $g_1^{\mathbf{w}} \approx f_3^{\mathbf{w}}$ by (SU1), but $g_1^{\mathbf{w}} \not\approx f_3^{\mathbf{w}}$ by (SU4).]

For $i = 1, 2, 3$, let $\mathscr{E}_i$ denote the set of elementary automorphisms E of $k[x_1, x_2, x_3]$ such that $E(x_j) = x_j$ for each $j \neq i$. We set $\mathscr{E} := \bigcup_{i=1}^{3} \mathscr{E}_i$.

Remark 1.1.15 If $(F, G) \in \mathscr{T}^2$ satisfies the Shestakov–Umirbaev condition, then we have the following:

(i) $G = FE_1E_2E_3$ for some $E_i \in \mathscr{E}_i$ by (SU1).
(ii) G is an elementary reduction of $F' := (g_1, g_2, f_3)$ by (SU1) and (SU5).
(iii) We have $\deg_{\mathbf{w}} F' \geq \deg_{\mathbf{w}} F$ by (SU2), but $\deg_{\mathbf{w}} G < \deg_{\mathbf{w}} F$ as shown later (Lemma 1.2.2 (i)).