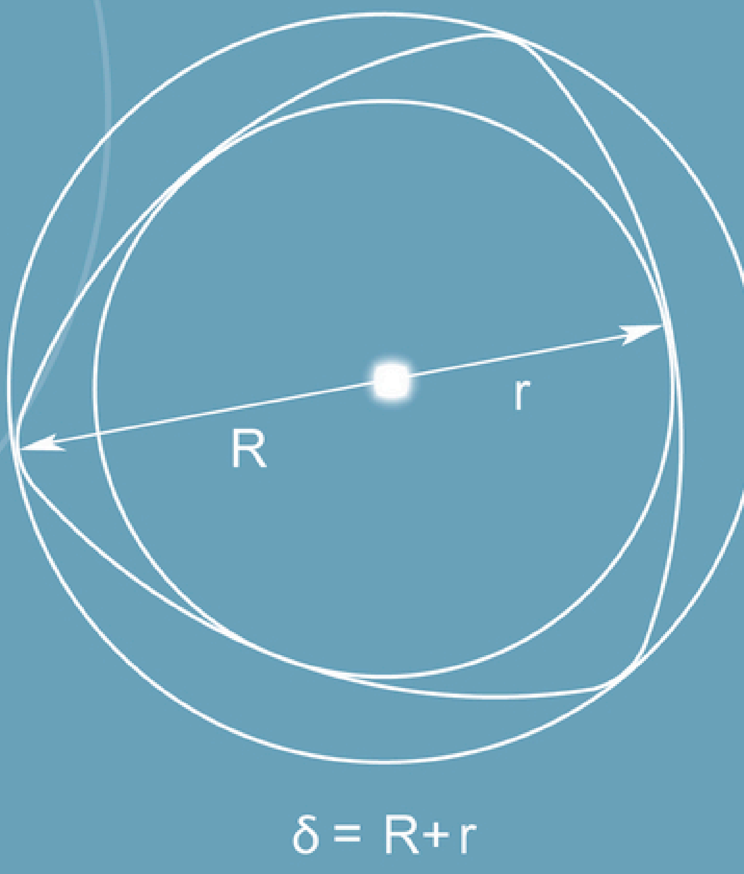
SOLUTIONS MANUAL TO ACCOMPANY

GEOMETRY OF CONVEX SETS

I. E. LEONARD
J. E. LEWIS



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PREFACE

These are the solutions to the odd numbered problems in the text *The Geometry of Convex Sets* by I. E. Leonard and J. E. Lewis.

Some of the solutions are from assignments we gave in class, some are not. In all of the solutions, we have provided details that added to the clarity and ease of understanding for beginning students, and when possible to the elegance of the solutions.

ED AND TED

Edmonton, Alberta, Canada
March 2016

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INTRODUCTION TO N-DIMENSIONAL GEOMETRY

1.2 POINTS, VECTORS, AND PARALLEL LINES

1.2.5 Problems

A remark about the exercises is necessary. Certain questions are phrased as statements to avoid the incessant use of "prove that". See Problem 1, for example. Such statements are supposed to be proved. Other questions have a "true–false" or "yes–no" quality. The point of such questions is not to guess, but to justify your answer. Questions marked with $*$ are considered to be more challenging. Hints are given for some problems. Of course, a hint may contain statements that must be proved.

1. Let S be a nonempty set in $\mathbb{R}^n$. If every three points of S are collinear, then S is collinear.

   Solution. Let x_1 and x_2 be two distinct points in S, then there is a unique line ℓ passing through these two points. Now let x be an arbitrary point in S, from the hypothesis, x_1, x_2, and x must be on some line ℓ', and since x_1 and x_2 uniquely determine the line ℓ, we must have $\ell' = \ell$. Therefore, every point x in S is on the line ℓ. $\square$

3. Given that the line L has the linear equation

$$\mu_1 x_1 + \mu_2 x_2 = \delta,$$

Solutions Manual to Accompany Geometry of Convex Sets, First Edition. I. E. Leonard and J. E. Lewis.

show that the point

$$\left(\frac{\mu_1\delta}{\mu_1^2+\mu_2^2}, \frac{\mu_2\delta}{\mu_1^2+\mu_2^2}\right)$$

is on the line, and that the vector $(-\mu_2, \mu_1)$ is parallel to the line.
Hint. If p is on the line and if $p+v$ is also on the line, then v must be parallel to the line.

Solution. Substituting the coordinates of this point into the linear equation for L, we see that

$$\mu_1 \cdot \frac{\mu_1\delta}{\mu_1^2+\mu_2^2} + \mu_2 \cdot \frac{\mu_2\delta}{\mu_1^2+\mu_2^2} = \frac{(\mu_1^2+\mu_2^2)\delta}{\mu_1^2+\mu_2^2} = \delta,$$

so that the given point is on L.

Since not both μ_1 and μ_2 are 0, we may assume that $\mu_1 \neq 0$, and let

$$p_1 = \left(\frac{\mu_1\delta}{\mu_1^2+\mu_2^2}, \frac{\mu_2\delta}{\mu_1^2+\mu_2^2}\right) \quad \text{and} \quad p_2 = \left(\frac{\delta}{\mu_1}, 0\right),$$

then both p_1 and p_2 are on the line L, and therefore $w = p_1 - p_2$ is parallel to L. However,

$$w = \frac{\mu_2\delta}{\mu_1(\mu_1^2+\mu_2^2)}\,(-\mu_2, \mu_1),$$

so the vector $v = (-\mu_2, \mu_1)$ is parallel to L.

Note that this follows immediately from the fact that the vector

$$v_\perp = (\mu_1, \mu_2)$$

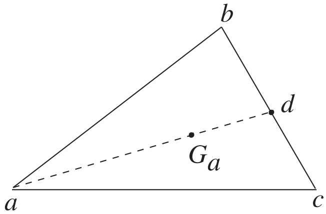
is the normal vector to the line L. $\square$

5. The centroid of three noncollinear points a, b, and c in $\mathbb{R}^n$ is defined to be

$$G = \frac{1}{3}(a+b+c).$$

Show that this definition of the centroid yields the synthetic definition of the centroid of the triangle with vertices a, b, c, namely, the point at which the three medians of the triangle intersect. Prove also that the medians do indeed intersect at a common point.

Solution. Given a triangle with vertices $a, b, c \in \mathbb{R}^n$, let $d \in \mathbb{R}^n$ be the midpoint of the segment $[b, c]$ and let $G_a \in \mathbb{R}^n$ be the point along the median ad which is $\frac{2}{3}$ the distance from a to d.



We have $G_a = a + \frac{2}{3}(d - a)$, and since $d = \frac{1}{2}(b + c)$, then

$$G_a = a + \frac{2}{3} \cdot \frac{1}{2}(b + c) - \frac{2}{3}a = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c.$$

If we define G_b and G_c similarly, then we see that

$$G_a = G_b = G_c = \frac{1}{3}(a + b + c),$$

so that the point $\frac{1}{3}(a + b + c)$ lies on each of the three medians. Thus, this is the synthetic definition of the centroid and the medians intersect at a single point. □

1.4 INNER PRODUCT AND ORTHOGONALITY

1.4.3 Problems

In the following exercises, assume that "distance" means "Euclidean distance" unless otherwise stated.

1. (a) The ***unit cube*** in $\mathbb{R}^n$ is the set of points

$$\{x = (\alpha_1, \alpha_2, \ldots, \alpha_n) : |\alpha_i| \leq 1, i = 1, 2, \ldots, n\}.$$

Draw the unit cube in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

(b) What is the length of the longest line segment that you can place in the unit cube of $\mathbb{R}^n$?

(c) What is the radius of the smallest Euclidean ball that contains the unit cube of $\mathbb{R}^n$?

Solution.

(a) The unit cubes are sketched below.