Solid Mechanics and Its Applications

M. B. Rubin

Continuum Mechanics with Eulerian Formulations of Constitutive Equations

Springer

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Volume 265

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ISSN 0925-0042 ISSN 2214-7764 (electronic)
Solid Mechanics and Its Applications
ISBN 978-3-030-57775-9 ISBN 978-3-030-57776-6 (eBook)
https://doi.org/10.1007/978-3-030-57776-6

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to my loving wife
Laurel and my children and (grandsons):
Adam & Dana (Leo & Tom); Daniel & Sefi.

Preface

My interest in mechanics was stimulated by my Scout Master Henry Layton when I was a Boy Scout. Henry was a mechanical engineer and a patent examiner who helped us build mini-bikes using bicycle parts and lawn mower engines. During my teenage years I bought a Craftsmen tool set at Sears and Roebuck, which I used to work on my cars and motorcycles. At the University of Colorado in Boulder, where I did my undergraduate degree in Mechanical Engineering, I learned that mathematics, vectors and tensors are the tools that I needed to truly understand the fundamentals of mechanics. Fortunately at Boulder, Prof. Frank Essenburg and Prof. William Wainwright helped me develop analytical skills and physical thinking needed to deepen my knowledge. They both encouraged me to continue my studies for a Ph.D. in applied mechanics after I graduated in December 1972.

I applied to the University of California at Berkeley and was accepted in the Department of Mechanical Engineering as a graduate student in applied mechanics. During my last semester at Boulder I took a course in continuum mechanics from a fluid mechanics professor who, unfortunately, really couldn't explain the deep physics of continuum mechanics. This caused me to change my major to bioengineering when I arrived at Berkeley for the fall quarter of 1973. However, I enrolled in a continuum mechanics course taught by Prof. Paul Naghdi who was clear, rigorous and explained the physical foundations. I thought then that if I studied bioengineering I would not know enough biology to formulate a problem and I would not know enough engineering to solve it. Consequently, I returned to applied mechanics and was truly fortunate to have Paul as a thesis advisor. Through my research, I have continued my interest in bioengineering. In my opinion, this interdisciplenary field requires experts from different fields to communicate and interact to make real progress.

Paul was a critical thinker who had the unique ability to read something that he had written as if he were an objective expert reading it for the first time. This talent helped him identify flaws in traditional approaches and create new ideas and formulations. My numerous discussions with Paul, both as a graduate student and as a colleague, challenged me and helped me develop as an independent researcher. I am immensely indebted to Paul for investing so much time to inspire and shape me as a researcher in continuum mechanics. Later I learned that both Frank Essenburg and William Wainwright were students of Paul so it is not surprising that I was attracted to Paul's lectures at Berkeley.

In August of 1979 I began work as a research engineer at SRI International. During my job interview I was told that as a theoretician I have to be willing to do experiments. At SRI, I was aided by a team of excellent technicians who taught me about many experimental problems as I acted as the supervisor of experiments. This exposure gave me a great appreciation for the difficulties of doing a good experiment, which has helped put a more physical perspective on my research over the years.

In October 1982 I moved to Israel with my wife Laurel to join the Faculty of Mechanical Engineering at Techion—Israel Institute of Technology, where I have spent my entire academic career, retiring as a Professor Emeritus in October 2019. I developed a friendship and working relationship with my senior colleague Prof. Sol Bodner, who was an experienced engineer with interests in both theory and experiments. My numerous discussions with Sol exposed me to the field of viscoplasticity and taught me how to think more physically about material response. I am also very much indebted to Sol for investing so much time in my development.

I have been teaching the course Introduction to Continuum Mechanics at Technion since the spring semester of 1983. The course and this book are based on the lecture notes of Paul Naghdi at Berkeley. Details of the presentation of this material have changed over the years as my understanding of continuum mechanics evolved due to my research and interactions with students, graduate students and colleagues, especially Prof. Eli Altus, with whom I had many discussions. During the first meeting of this course, I tell the students that continuum mechanics is a deep subject and that I am still learning after having been an active researcher in continuum mechanics for over 40 years. In my opinion, continuum mechanics is a theoretical umbrella for almost all of engineering because the thermomechanical theory applies to a broad range of solid materials (elastic, elastic–inelastic, elastic–viscoelastic) and fluid materials (gases, inviscid, viscous and viscoelastic liquids). Continuum mechanics provides a theoretical framework to ensure that we don't make fundamental blunders. However, the true beauty of the field is that we will always be challenged to use our theoretical expertise and physical intuition to synthesize experimental data to propose functional forms for constitutive equations that describe new important features of material response that needs to be modeled.

My experience has also been enriched by having been a regular Visiting Faculty at Lawrence Livermore National Laboratory (LLNL) since 1985. Dr. Lewis Glenn and Dr. Willy Moss were my first boss and colleague, respectively, at LLNL. Over the years I have had the opportunity to work with a number of very talented researchers at LLNL who have contributed to some of the constitutive equations presented in this book. At LLNL, I was exposed to the field of shock physics in geological materials which challenged me to develop specific functional forms for strongly coupled thermomechanical response that can be used to match experimental data. The exposure to real problems and the ability to work with excellent computational mechanics people at LLNL has enriched my ability to think physically. Often I would have a number of ideas why the simulations using the constitutive equations for a particular material do not match experimental data. In working with my colleagues at LLNL I realized that it is important to find the simplest way to "hack" the computer code to test an idea to see if it really makes a difference. Once the ideal that makes a difference has been identified, then it is necessary to develop the constitutive equations rigorously. It remains a challenge to ensure that the "hack" is removed and the rigorous equations have been programed.

In addition, at LLNL I learned the importance of numerical algorithms. This has particular relevance for the development of constitutive equations. Theoreticians can often propose different functional forms which model the same limiting cases. When working with computational mechanics it is important to choose those functional forms for modeling a specific material response which have the correct limits but also simplify the numerical algorithm.

I am also indebted to my colleague and friend Prof. Mahmood Jabareen in the Faculty of Civil and Environmental Engineering at Technion. His computational mechanics expertise was essential for the transition of the Cosserat Point Element (CPE) technology from a theoretical concept that I proposed in 1985 to algorithms that have been implemented in the commercial computer code LS-DYNA. We also collaborated on a number of papers which have shaped some of the ideas presented in this book, especially those on physical orthotropic invariants, the formulation of constitutive equations with a smooth elastic–inelastic transition and growth of biological materials. My graduate student and Post-Doctoral Fellow Dr. Mahmoud Safadi learned computational mechanics from Prof. Jabareen which was essential for his successful implementation of the constitutive equations for growth in the commercial computer code Abaqus. His expertise was used for simulations in our joint papers that highlighted the importance of the Eulerian formulation for growth. In addition, discussions with Dr. Gal Shmuel and Prof. Reuven Segev helped improve the presentation of the notions of a uniform material, a homogeneous material and a uniform material state. Prof. Roger Fosdick and Prof. Albrecht Bertram provided constructive criticism that improved the presentation of invariance under Superposed Rigid Body Motions, especially for constrained materials. Also, my wife Laurel proofread this book which helped identify and correct a number of typographical errors.

I am certain that engineers are essential to the future of Israel. Therefore, I am honored to be a Professor Emeritus from Technion, which is the best engineering university in Israel. Having taught at Technion makes me feel that I have contributed to the future of Israel through students who have been influenced by my teaching. In particular, I derive great satisfaction knowing that some of my graduate students have made significant contributions to the security and economic development of Israel. I am sure that I could not attain such personal satisfaction in my profession having been a professor anywhere else in the world.

I would also like to acknowledge the German Israel Foundation (GIF), the Israel Science Foundation (ISF) and my Gerard Swope Chair in Mechanics which provided financial support over the years.

My research on large deformation inelasticity and on growth of biological materials has caused me to develop an Eulerian formulation of constitutive equations for elastic and inelastic response. The important physical feature of the Eulerian formulation is that it removes arbitrariness of the choice of: a reference configuration, an intermediate zero-stress configuration, a total deformation measure and an inelastic deformation measure. The main new features of this book are the discussion of the importance of the Eulerian formulation and the demonstration of how it can be used to develop a broad range of specific constitutive equations in thermomechanics.

Haifa, Israel M. B. Rubin

Contents

Chapter 1
Introduction

Abstract The objective of this introductory chapter is to present an overview of the contents of this book and to discuss the importance of Eulerian formulations of constitutive equations. Specifically, simple one-dimensional examples are used to identify unphysical arbitrariness in the classical Lagrangian formulations of constitutive equations that can and should be removed.

1.1 Content of the Book

Continuum mechanics is concerned with the fundamental equations that describe the nonlinear thermomechanical response of all deformable media. Throughout this book, attention is limited to a simple material whose constitutive response does not depend on higher order gradients of deformation. Although the constitutive equations are phenomenological and are proposed to model the macroscopic response of materials, they are reasonably accurate for many studies of micro- and nano-mechanics where the typical length scales approach, but are still larger than, those of individual atoms. In this sense, the general thermomechanical theory provides a theoretical umbrella for most areas of study in mechanical engineering. In particular, continuum mechanics includes as special cases theories of: solids (elastic, inelastic, viscoelastic, etc.), fluids (compressible, incompressible, viscous) and the thermodynamics of heat conduction including dissipation due to inelastic effects.

A number of books have been written which discuss the fundamentals of continuum mechanics [5, 7, 9, 11], the theory of elasticity [1, 12], the theory of plasticity [2, 3] as well as the thermomechanical theory [18]. The new aspect of this book is its emphasis on an Eulerian formulation of constitutive equations for elastic materials, elastic–inelastic materials and growing biological tissues. The standard Lagrangian formulation of constitutive equations and the need for an Eulerian formulation will be discussed in detail from a physical point of view and specific constitutive equations will be described for different classes of materials.

Apart from this introduction, the material in this book on continuum mechanics is divided into five chapters. Chapter 2 develops a basic knowledge of tensor

M. B. Rubin, *Continuum Mechanics with Eulerian Formulations of Constitutive Equations*, Solid Mechanics and Its Applications 265,
https://doi.org/10.1007/978-3-030-57776-6_1

analysis using both indicial notation and direct notation. Although tensor operations in general curvilinear coordinates are needed to express spatial derivatives like those in the gradient and divergence operators, these special operations required to translate quantities in direct notation to component forms in special coordinate systems are merely mathematical in nature. Moreover, details of general curvilinear tensor analysis unnecessarily complicate the presentation of the fundamental physical issues in continuum mechanics. Consequently, here attention is mainly restricted to tensors expressed in terms of constant rectangular Cartesian base vectors to simplify the discussion of spatial derivatives and to concentrate on the main physical issues. However, an introduction to tensors with respect to curvilinear coordinates is presented in Appendix F.

Chapter 3 develops tools to analyze nonlinear deformation and motion of continua. Specifically, measures of deformation and their rates are introduced. Also, the group of Superposed Rigid Body Motions (SRBM) is introduced for later fundamental analysis of invariance of constitutive equations under SRBM.

Chapter 4 develops the balance laws that are applicable for simple continua, which are characterized by local measures of deformation. The notion of the stress tensor and its relationship to the traction vector is developed. Local forms of the equations of motion are derived from the global forms of the balance laws. Referential forms of the equations of motion are discussed and the relationships between different stress measures are developed for completeness, but they are not used in the Eulerian formulation of constitutive equations. Also, invariance under SRBM of the balance laws and the kinetic quantities are discussed.

Chapter 5 presents an introduction to constitutive theory. Although there is general consensus on the kinematics of continua, the notion of constitutive equations for special materials remains an active area of research in continuum mechanics. Specifically, in these sections the theoretical structure of constitutive equations for nonlinear anisotropic elastic solids, isotropic elastic solids, viscous and inviscid fluids, viscous dissipation, elastic–inelastic solids and viscoelastic solids are discussed.

Chapter 6 describes thermomechanical processes and the fundamental balance laws and restrictions of second laws of thermomechanics that control these processes. In addition, specific constitutive equations for: thermoelastic materials, thermoelastic–inelastic materials, orthotropic thermoelastic–inelastic materials, shock waves, porous materials and growing biological tissues are discussed. Also, jump conditions for the thermomechanical balance laws are developed.

1.2 Comparison of the Lagrangian and Eulerian Formulations

Unphysical arbitrariness of the choices of: the reference configuration; a zero-stress intermediate configuration; a total deformation measure and a plastic deformation measure has been discussed in a series of papers [15–17]. To simplify the discussion

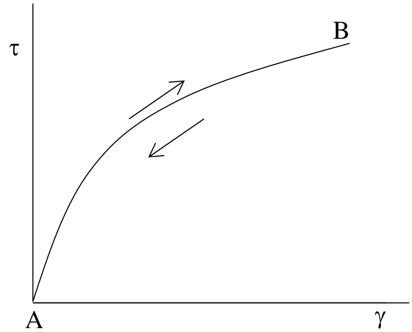

Fig. 1.1 Response of a homogeneous nonlinear elastic material to homogeneous proportional loading in shear from a uniform zero-stress material state A to a uniform loaded material state B with unloading along the same path back to the same uniform zero-stress material state A

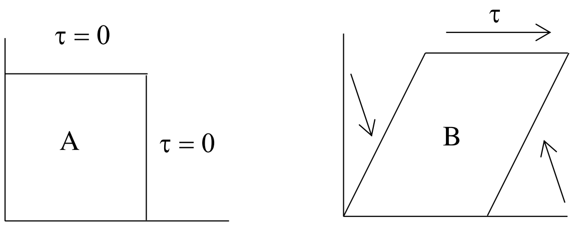

Fig. 1.2 Sketch of the deformation of a homogeneous nonlinear elastic material subjected to homogenous proportional loading in shear from a uniform zero-stress material state A to a uniform loaded material state B with unloading along the same path back to the same uniform zero-stress material state A

of these issues, here attention is limited to the purely mechanical theory at constant zero-stress reference temperature.

Figure 1.1 shows the shear stress τ versus the total shear strain γ for a homogeneous nonlinear elastic material subjected to homogeneous proportional loading from a uniform zero-stress material state A to a uniform loaded material state B with unloading along the same path to the same uniform zero-stress material state A. Figure 1.2 shows a sketch of the associated deformations.

These figures exhibit the property that a homogeneous nonlinear elastic material in a uniform zero-stress material state, which is loaded to a deformed state, will return to its zero-stress shape and volume when unloaded. In this sense the nonlinear elastic material remembers its zero-stress shape and density. This also suggests that the response of a homogeneous nonlinear elastic material can be characterized by a Lagrangian formulation of the constitutive equation in terms of a Lagrangian strain that measures deformations from a reference configuration with a uniform stress-free material state and vanishes in this reference configuration.

Figure 1.3 shows the shear stress τ versus the total shear strain γ for a homogeneous nonlinear elastic–plastic material subjected to homogeneous proportional

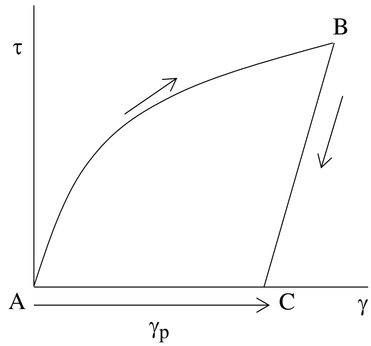

Fig. 1.3 Response of a homogeneous nonlinear elastic–plastic material to homogeneous proportional loading in shear from a uniform zero-stress material state *A* to a uniform loaded material state *B* with unloading along a different path to a uniform zero-stress material state *C* with a residual total strain γ_p

loading from a uniform zero-stress material state *A* to a uniform loaded material state *B* with unloading along a different path to a uniform zero-stress material state *C* with a residual total strain γ_p. Figure 1.4 shows a sketch of the associated deformations. Motivated by the Lagrangian formulation of elastic response, in addition to the Lagrangian total strain from the reference configuration, it is common to introduce a plastic deformation (see γ_p in Fig. 1.3) measured from the reference configuration to the uniform zero-stress intermediate configuration (see state *C* in Fig. 1.3). Also, it is common to define an elastic deformation measure in terms of the total and plastic deformation measures. In this sense, the plastic deformation measure is a history-dependent variable that is determined by integrating an evolution equation for its time rate of change.

Onat [13] discussed physical restrictions on internal state variables. This discussion proposed that internal state variables, which are determined by integrating evolution equations over time, are specified to measure properties of the material response that define the current state of the material. Moreover, since these evolution equations need initial conditions it is necessary that the values of the internal state variables be, in principle, measurable directly or indirectly by experiments on multiple identical samples of the material in its current material state. In this sense, the material state must be characterized by internal state variables whose values are measurable in the current state.

From this perspective, it is necessary to ask if the deformation measures that are used to characterize material response are acceptable internal state variables. For a homogeneous elastic material, it is common to define the deformation gradient tensor **F** from a uniform stress-free reference configuration to the current deformed configuration to characterize the constitutive equation of the elastic material. For this elastic material, it follows that since the volume and shape of the material are unique in any zero-stress material state, **F** is only known to within an arbitrary proper orthogonal rotation tensor **R** in any zero-stress material state. This means that the zero-stress value of **F** has arbitrariness due to three orientation angles associated with **R** which cannot be determined by experiments on identical samples of the same material in the current configuration. Consequently, **F** is not an acceptable internal state variable in the sense discussed by Onat [13]. For this reason, **F** should not

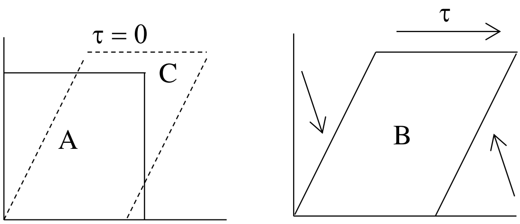

Fig. 1.4 Sketch of the deformation of a homogeneous nonlinear elastic–plastic material subjected to homogenous proportional loading in shear from a uniform zero-stress material state *A* to a uniform loaded material state *B* with unloading along a different path to a uniform zero-stress material state *C* (dashed lines) with a different shape from that in the uniform zero-stress material state *A*

appear in any constitutive equation for material response, even for nonlinear elastic materials. However, for the solution of a specific problem it is often convenient to parameterize the solution using the total deformation gradient $\mathbf{F}$ from a known specified reference configuration. In this sense, it is important to distinguish between a tensorial measure of elastic deformation from a zero-stress material state and the total deformation gradient from a specified reference configuration.

The use of $\mathbf{F}$ in constitutive equations for elastic–plastic materials is even more problematic physically. Even if plastic deformations are isochoric, a homogeneous elastic–plastic material that is loaded from a uniform zero-stress material state has no unique shape in another uniform zero-stress material state (see the initial state *A* and the intermediate state *C* in Fig. 1.4). This means that only the volumetric part of $\mathbf{F}$ can be determined in a uniform zero-stress material state so there are eight degrees of arbitrariness in $\mathbf{F}$, three associated with orientation changes and five associated with distortional deformations. The following statement by Gilman in the discussion section in [8] refers to this physical arbitrariness.

> It seems very unfortunate to me that the theory of plasticity was ever cast into the mold of stress–strain relations because ‘strain’ in the plastic case has no physical meaning that is related to the material of the body in question. It is rather like trying to deduce some properties of a liquid from the shape of the container that holds it. The plastic behavior of a body depends on its structure (crystalline and defect) and on the system of stresses that is applied to it. The structure will vary with plastic strain, but not in a unique fashion. The variation will also depend on the initial structure, the values of whatever stresses are applied, and on time (some recovery occurs in almost any material at any temperature).

The Eulerian formulation of constitutive equations discussed in this book and in [15–17] is motivated by the work of Eckart [4] for elastic–inelastic solids, by Leonov [10] for polymeric liquids and is based on the work in [14]. This Eulerian formulation uses evolution equations for the material time derivative of internal state variables. More specifically, the formulation is considered to be Eulerian because the evolution equations depend only on quantities that can, in principle, be measured in the current state of the material. It will be shown that this Eulerian formulation removes arbitrariness of the choice of: the reference configuration; an intermediate configuration; a total deformation measure and an inelastic deformation measure.

Table 1.1 Comparison of the Lagrangian (Classical) and Eulerian (Eckart) formulations

Lagrangian (Classical)	Eulerian (Eckart)
$\sigma = E\varepsilon_e$	$\sigma = E\varepsilon_e$
$\dot{\varepsilon} = \partial v/\partial x$ $\dot{\varepsilon}_p = \frac{\Gamma}{E}\,\sigma$ $\varepsilon_e = \varepsilon - \varepsilon_p$	$\dot{\varepsilon}_e = \partial v/\partial x - \Gamma\,\varepsilon_e$
$\varepsilon(0) = ?$ $\varepsilon_p(0) = ?$	$\varepsilon_e(0) = \frac{\sigma(0)}{E}$

Table 1.1 records the basic equations needed to compare the differences between the Lagrangian (Classical) formulation and the Eulerian (Eckart) formulation for inelasticity using a simple one-dimensional model. In this model the strains are small so the notion of Lagrangian is used for quantities that are referred to a reference configuration. Specifically, the axial stress σ is determined by the axial elastic strain ε_e using Young's modulus of elasticity E in both formulations. However, in the Lagrangian formulation, it is necessary to define the total axial strain ε, the plastic or inelastic axial strain ε_p, as well as the axial elastic strain ε_e. Specifically, the total strain ε is determined by integrating an evolution equation in terms of the velocity gradient $\partial v/\partial x$. The inelastic strain ε_p is determined by integrating an evolution equation in terms of the stress σ and a non-negative function Γ that controls inelastic deformation rate, and the elastic strain ε_e is defined by the difference between the total strain and the inelastic strain. In contrast, in the Eulerian formulation the elastic strain ε_e is determined directly by integrating an evolution equation in terms of the velocity gradient $\partial v/\partial x$, the elastic strain ε_e and the function Γ.

The Eulerian evolution equation for elastic strain ε_e is consistent with the equation in the Lagrangian formulation and can be obtained by taking the time derivative $\dot{(\ )}$ of the algebraic expression for ε_e and replacing $\dot{\varepsilon}$ and $\dot{\varepsilon}_p$ with their evolution equations. However, the physics of these two formulations are different. In the Lagrangian formulation it is necessary to specify the initial values $\varepsilon(0)$ and $\varepsilon_p(0)$. But these quantities are both referred to an arbitrary choice of the reference configuration. This can be made explicit by noting that the same initial value $\varepsilon_e(0)$ of elastic strain can be obtained by changing the reference configuration using the arbitrary value A, such that

$$\varepsilon_e(0) = \varepsilon(0) - \varepsilon_p(0) = [\varepsilon(0) - A] - [\varepsilon_p(0) - A]\,, \tag{1.2.1}$$

where the scalar A in this one-dimensional model characterizes the influence of the arbitrariness of the reference configuration in a similar manner to the tensor $\mathbf{A}$ in the nonlinear three-dimensional theory discussed in (5.11.24). This arbitrariness means that the individual initial values $\varepsilon(0)$ and $\varepsilon_p(0)$ needed to integrate the evolution equations for ε and ε_p cannot be measured independently. Consequently, ε and ε_p are not internal state variables in the sense of Onat [13].

In contrast to the Lagrangian formulation, in the Eulerian formulation the elastic strain ε_e is introduced directly through an evolution equation for its rate and the initial value $\varepsilon_e(0)$ needed to integrate this equation can be determined by the measuring the initial value $\sigma(0)$ of stress. Consequently, the elastic strain ε_e is an internal state variable in the sense of Onat [13] since it is measurable. Moreover, the arbitrariness associated with the orientation of the body in a zero-stress intermediate configuration in the three-dimensional theory discussed in (5.11.25) cannot be analyzed using the simple one-dimensional model.

Following the work of Eckart [4] and Leonov [10], an Eulerian formulation for elastically isotropic inelastic materials introduces a symmetric positive-definite elastic deformation tensor $\mathbf{B}_e$ through an evolution equation for its material time derivative. Moreover, using the work of Flory [6], $\mathbf{B}_e$ is expressed in terms of the elastic dilatation J_e and the symmetric positive-definite unimodular elastic distortional deformation tensor $\mathbf{B}'_e$ defined by

$$J_e = \sqrt{\det \mathbf{B}_e}\,, \quad \mathbf{B}'_e = J_e^{-2/3}\mathbf{B}_e\,, \quad \det \mathbf{B}'_e = 1\,. \tag{1.2.2}$$

Then, for elastically isotropic thermoelastic–inelastic materials evolution equations are proposed directly for J_e and $\mathbf{B}'_e$, and the Helmholtz free energy ψ per unit mass and the Cauchy stress $\mathbf{T}$ are specified by constitutive equations which depend on $J_e, \mathbf{B}'_e$ and the absolute temperature θ in the forms

$$\psi = \psi(J_e, \mathbf{B}'_e, \theta)\,, \quad \mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e, \theta)\,. \tag{1.2.3}$$

This constitutive equation for stress is restricted to be invertible with J_e and $\mathbf{B}'_e$ admitting the representations

$$J_e = J_e(\mathbf{T}, \theta)\,, \quad \mathbf{B}'_e = \mathbf{B}'_e(\mathbf{T}, \theta)\,. \tag{1.2.4}$$

The constitutive equation for stress is further restricted so that a zero-stress material state at zero-stress reference temperature θ_z requires

$$J_e(0, \theta_z) = 1\,, \quad \mathbf{B}'_e(0, \theta_z) = \mathbf{I}\,, \tag{1.2.5}$$

where $\mathbf{I}$ is the second-order identity tensor. These restrictions ensure that J_e and $\mathbf{B}'_e$ are internal state variables in the sense of Onat [13] since their initial values required to integrate their evolution equations can be determined by the measured values of $\mathbf{T}$ and θ in the initial state of the material.

Another specific example where it is clear that it is not sufficient to formulate constitutive equations in terms of a Lagrangian deformation measure is an anisotropic elastic material with a quadratic strain energy function. Specifically, let $\mathbf{E}$ be a Lagrangian total strain measure from a reference configuration with a uniform zero-stress material state and consider the quadratic strain energy function Σ per unit mass specified by

$$\rho_z \Sigma = \frac{1}{2} \mathbf{K} \cdot \mathbf{E} \otimes \mathbf{E}, \tag{1.2.6}$$

where ρ_z is a constant zero-stress mass density, $\mathbf{K}$ is a constant fourth-order stiffness tensor, $\otimes$ is the tensor product operator and $(\cdot)$ is the inner product between two tensors of any order. Referring these tensors to an arbitrary rectangular Cartesian orthonormal triad of vectors $\mathbf{e}_i$ in the reference configuration yields the expression

$$\rho_z \Sigma = \frac{1}{2} K_{ijkl} E_{ij} E_{kl}. \tag{1.2.7}$$

For a general anisotropic elastic material K_{ijkl} has the symmetries

$$K_{jikl} = K_{ijlk} = K_{klij} = K_{ijkl}, \tag{1.2.8}$$

so it is characterized by 21 independent material constants. Although this quadratic strain energy function can model general anisotropic elastic response, the representation is incomplete since it is necessary to connect the components K_{ijkl} of the stiffnesses tensor with identifiable material directions.

Following the work in [14], the elastic deformations and material orientations in the Eulerian formulation for elastically anisotropic materials discussed in this book are characterized by a right-handed triad of linearly independent microstructural vectors $\mathbf{m}_i$ with the elastic dilatation J_e defined by

$$J_e = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 > 0. \tag{1.2.9}$$

Also, the elastic metric m_{ij} is defined by

$$m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j. \tag{1.2.10}$$

Then, for elastically anisotropic thermoelastic–inelastic materials evolution equations are proposed directly for $\mathbf{m}_i$, and the Helmholtz free energy ψ per unit mass and the Cauchy stress $\mathbf{T}$ are specified by constitutive equations which depend on $\mathbf{m}_i$ and the absolute temperature θ in the forms

$$\psi = \psi(m_{ij}, \theta), \quad \mathbf{T} = \mathbf{T}(\mathbf{m}_i, \theta). \tag{1.2.11}$$

This constitutive equation for stress is restricted to be invertible with $\mathbf{m}_i$ admitting the representations

$$\mathbf{m}_i = \mathbf{m}_i(\mathbf{T}, \theta). \tag{1.2.12}$$

The constitutive equation for stress is further restricted so that a zero-stress material state at zero-stress reference temperature θ_z requires

$$m_{ij}(0, \theta_z) = \mathbf{m}_i(0, \theta_z) \cdot \mathbf{m}_j(0, \theta_z) = \delta_{ij}, \quad J_e(0, \theta_z) = 1, \tag{1.2.13}$$

where δ_{ij} is the Kronecker delta and $\mathbf{m}_i$ have been defined to be orthonormal in this zero-stress material state at zero-stress reference temperature. These restrictions ensure that $\mathbf{m}_i$ are internal state variables in the sense of Onat [13] since their initial values required to integrate their evolution equations can be determined by the measured values of $\mathbf{T}$ and θ in the initial state of the material. Depending on the material being modeled it may be necessary to consider the response of identical samples of the material in its current state to different loading paths to determine the values of $\mathbf{m}_i$ in the current state. Further in this regard, it is noted that symmetries of the material response characterized by the Helmholtz free energy ψ make the response of the material insensitive to any indeterminacy in the inversion (1.2.12) for $\mathbf{m}_i$.

This representation has the advantage that the indices $i = 1, 2, 3$ of these vectors characterize specific material directions. It will be shown that these microstructural vectors can be used to model elastic deformations for anisotropic elastic materials and for the rate-independent and rate-dependent response of anisotropic elastic–inelastic materials.

Details of fundamental aspects of the Eulerian formulation of constitutive equations can be found in Sects. 3.11, 3.14, 5.2, 5.3, 5.4, 5.11, 5.12 and in Chap. 6 for thermomechanical response.

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Chapter 2
Basic Tensor Analysis

Abstract Tensors are mathematical objects which ensure that mathematical equations characterizing physics are insensitive to arbitrary choices of a coordinate system. The objective of this chapter is to present a review of tensor analysis using both index and direct notations. To simplify the presentation of tensor calculus, attention is limited to tensors expressed relative to fixed rectangular Cartesian base vectors. (Some of the content in this chapter has been adapted from Rubin (Cosserat theories: shells, rods and points. Springer Science & Business Media, Berlin, 2000) with permission.)

2.1 Vector Algebra

Tensors, tensor algebra and tensor calculus are needed to formulate physical equations in continuum mechanics which are insensitive to arbitrary choices of coordinates. To understand the mathematics of tensors it is desirable to start with the use of a language called indicial notation which develops simple rules governing these tensor manipulations. For the purposes of describing this language it is convenient to introduce a fixed right-handed triad of orthonormal rectangular Cartesian base vectors denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. From the study of linear vector spaces, it is recalled that vectors satisfy certain laws of addition and multiplication by a scalar. Specifically, if $\mathbf{a}$ and $\mathbf{b}$ are vectors then the quantity

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{2.1.1}$$

is a vector defined by the parallelogram law of addition. Furthermore, recall that the operations

M. B. Rubin, *Continuum Mechanics with Eulerian Formulations of Constitutive Equations*, Solid Mechanics and Its Applications 265,
https://doi.org/10.1007/978-3-030-57776-6_2

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= \mathbf{b}+\mathbf{a} && \text{(commutative law)}\,,\\
(\mathbf{a}+\mathbf{b})+\mathbf{c} &= \mathbf{a}+(\mathbf{b}+\mathbf{c}) && \text{(associative law)}\,,\\
\alpha\mathbf{a} &= \mathbf{a}\alpha && \text{(multiplicative law)}\,,\\
\mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a} && \text{(commutative law)}\,,\\
\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} && \text{(distributive law)}\,,\\
\alpha(\mathbf{a}\cdot\mathbf{b}) &= (\alpha\mathbf{a})\cdot\mathbf{b} && \text{(distributive law)}\,,\\
\mathbf{a}\times\mathbf{b} &= -\mathbf{b}\times\mathbf{a} && \text{(lack of commutativity)}\,,\\
\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} && \text{(distributive law)}\,,\\
\alpha(\mathbf{a}\times\mathbf{b}) &= (\alpha\mathbf{a})\times\mathbf{b} && \text{(associative law)}
\end{aligned} \tag{2.1.2}$$

are satisfied for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and all real numbers α, where $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar product (or dot product) and $\mathbf{a}\times\mathbf{b}$ denotes the vector product (or cross product) between the vectors $\mathbf{a}$ and $\mathbf{b}$.

The Scalar Triple Product

The scalar triple product of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ has the property that the dot and cross products can be interchanged

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c}\,. \tag{2.1.3}$$

Moreover, using the results

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = -\mathbf{b}\times\mathbf{a}\cdot\mathbf{c} = -\mathbf{b}\cdot\mathbf{a}\times\mathbf{c} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = \mathbf{c}\times\mathbf{a}\cdot\mathbf{b}\,, \tag{2.1.4}$$

it follows that the order of the vectors in the scalar triple product can be permuted

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = \mathbf{c}\times\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\times\mathbf{c}\cdot\mathbf{a}\,. \tag{2.1.5}$$

The Vector Triple Product

The vector triple product of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ can be expanded to obtain

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,. \tag{2.1.6}$$

To prove this result it is noted that this vector must be perpendicular to both $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. But $\mathbf{b}\times\mathbf{c}$ is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$ so the vector triple product must be a vector in the plane of $\mathbf{b}$ and $\mathbf{c}$. Moreover, the vector triple product is linear in the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The expression (2.1.6) can be checked by considering the special case of $\mathbf{a}=\mathbf{e}_1$, $\mathbf{b}=\mathbf{e}_3$ and $\mathbf{c}=\mathbf{e}_1$.

2.2 Indicial Notation

Quantities written in indicial notation have a finite number of indices attached to them. Since the number of indices can be zero, a quantity with no index can also be considered to be written in indicial notation. The language of indicial notation is quite simple because only two types of indices can appear in any term. Either the index is a free index or it is a repeated index. Also, a simple summation convention is defined which applies only to repeated indices. These two types of indices and the summation convention are defined as follows.

Free Indices:
Indices that appear only once in a given term are known as free indices. In this regard, a term in an equation is a quantity that is separated by a plus, minus or equal sign. Here, each of these free indices will take the values $(1, 2, 3)$. For example, i is a free index in each of the following expressions

$$(x_1, x_2, x_3) = x_i \quad (i = 1, 2, 3)\,, \tag{2.2.1a}$$

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbf{e}_i \quad (i = 1, 2, 3)\,. \tag{2.2.1b}$$

Repeated Index:
Indices that appear twice in a given term are known as a repeated index. For example, i and j are free indices and m and n are repeated indices in the following expressions

$$a_i b_j c_m T_{mn} d_n\,, \quad A_{immjnn}\,, \quad A_{imn} B_{jmn}\,. \tag{2.2.2}$$

It is important to emphasize that in the language of indicial notation an index *can never appear more than twice* in any term.

Einstein Summation Convention:
When an index appears as a repeated index in a term that index is understood to take on the values $(1, 2, 3)$ and the resulting terms are summed. Thus, for example,

$$x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3\,. \tag{2.2.3}$$

Because of this summation convention, a repeated index is also known as a dummy index since its replacement by any other letter not appearing as a free index and also not appearing as another repeated index does not change the meaning of the term in which it occurs. For examples,

$$x_i \mathbf{e}_i = x_j \mathbf{e}_j\,, \quad a_i b_m c_m = a_i b_j c_j\,. \tag{2.2.4}$$

It is important to emphasize that the same free indices must appear in each term in an equation so that for example the free index i in $(2.2.4)_2$ must appear on each side of the equality.

Kronecker Delta:
The Kronecker delta δ_{ij} is defined by

$$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{bmatrix} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{bmatrix} . \tag{2.2.5}$$

Since the Kronecker delta δ_{ij} vanishes unless $i = j$ it exhibits the following exchange property

$$\delta_{ij} x_j = (\delta_{i1} x_1,\ \delta_{i2} x_2,\ \delta_{i3} x_3) = (x_1, x_2, x_3) = x_i . \tag{2.2.6}$$

Notice that the Kronecker delta can be removed by replacing the repeated index j in (2.2.6) by the free index i.

Recalling that an arbitrary vector $\mathbf{a}$ in Euclidean 3-Space can be expressed as a linear combination of the base vectors $\mathbf{e}_i$ it can be expressed in the form

$$\mathbf{a} = a_i \mathbf{e}_i . \tag{2.2.7}$$

Consequently, it follows that the components a_i of $\mathbf{a}$ can be calculated using the Kronecker delta

$$a_i = \mathbf{e}_i \cdot \mathbf{a} = \mathbf{e}_i \cdot (a_m \mathbf{e}_m) = (\mathbf{e}_i \cdot \mathbf{e}_m) a_m = \delta_{im} a_m = a_i . \tag{2.2.8}$$

Notice that when the expression (2.2.7) for $\mathbf{a}$ is substituted into (2.2.8) it is necessary to change the repeated index i in (2.2.7) to another letter m because the letter i already appears in (2.2.8) as a free index. It also follows that the Kronecker delta can be used to calculate the dot product between two vectors $\mathbf{a}$ and $\mathbf{b}$ with components a_i and b_i, respectively, by

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i (\mathbf{e}_i \cdot \mathbf{e}_j) b_j = a_i \delta_{ij} b_j = a_i b_i . \tag{2.2.9}$$

Permutation Symbol:
The permutation symbol ε_{ijk} is defined by

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = \begin{bmatrix} 1 & \text{if } (i, j, k) \text{ are an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ are an odd permutation of } (1,2,3) \\ 0 & \text{if at least two of } (i, j, k) \text{ have the same value} \end{bmatrix} . \tag{2.2.10}$$

This definition suggests that the permutation symbol can be used to calculate the vector product between two vectors. To this end, it will be shown that

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k . \tag{2.2.11}$$

Proof Since $\mathbf{e}_i \times \mathbf{e}_j$ is a vector in Euclidean 3-Space for each choice of the indices i and j, it follows that it can be represented as a linear combination of the base vectors $\mathbf{e}_k$ such that

$$\mathbf{e}_i \times \mathbf{e}_j = A_{ijk}\mathbf{e}_k \,, \tag{2.2.12}$$

where the components A_{ijk} need to be determined. In particular, taking the dot product of (2.2.12) with $\mathbf{e}_k$ and using the definition (2.2.10) yields

$$\varepsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = A_{ijm}\mathbf{e}_m \cdot \mathbf{e}_k = A_{ijm}\delta_{mk} = A_{ijk} \,, \tag{2.2.13}$$

which proves the result (2.2.11). Now using (2.2.11), it follows that the vector product between the vectors $\mathbf{a}$ and $\mathbf{b}$ can be represented in the form

$$\mathbf{a} \times \mathbf{b} = (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) = (\mathbf{e}_i \times \mathbf{e}_j)a_ib_j = \varepsilon_{ijk}a_ib_j\mathbf{e}_k \,. \tag{2.2.14}$$

Additional Properties of the Permutation Symbol:
Using (2.1.3) and (2.1.6) it can be shown that

$$\begin{aligned} &\varepsilon_{ijk}\varepsilon_{rsk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot (\mathbf{e}_r \times \mathbf{e}_s) = \mathbf{e}_i \cdot [\mathbf{e}_j \times (\mathbf{e}_r \times \mathbf{e}_s)] = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \,, \\ &\varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir} \,, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6 \,. \end{aligned} \tag{2.2.15}$$

Also, recall that the determinant of a matrix M_{ij} can be expressed in the forms

$$\begin{aligned} &\det(M_{mn}) = \varepsilon_{ijk}M_{i1}M_{j2}M_{k3} \,, \\ &\varepsilon_{rst}\det(M_{mn}) = \varepsilon_{ijk}M_{ir}M_{js}M_{kt} \,, \\ &\det(M_{mn}) = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{rst}M_{ir}M_{js}M_{kt} \,. \end{aligned} \tag{2.2.16}$$

Contraction:
Contraction is the process of replacing two free indices in a given expression with the same index together with the implied summation convention. For example, contraction on the free indices i, j in δ_{ij} yields

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \,. \tag{2.2.17}$$

Note that contraction on the set of $9 = 3^2$ quantities T_{ij} can be performed by multiplying T_{ij} by δ_{ij} to obtain

$$T_{ij}\delta_{ij} = T_{ii} \,. \tag{2.2.18}$$

2.3 Direct Notation (Special Case)

A scalar is sometimes referred to as a zero-order tensor and a vector is sometimes referred to as a first-order tensor. Higher order tensors are defined inductively starting with the notion of a first-order tensor or vector. Specifically, since a second-order tensor is a linear operator whose domain is the space of all vectors and whose range is the space of all vectors it is possible to define the second-order tensor inductively using vector spaces.

Tensor of Order M:
The quantity $\mathbf{T}$ is called a tensor of order M ($M \geq 2$) if it is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{Tv}$ or $\mathbf{vT}$ is a tensor of order $M-1$. Since $\mathbf{T}$ is a linear operator it satisfies the following rules

$$\mathbf{T}(\mathbf{v}+\mathbf{w}) = \mathbf{Tv}+\mathbf{Tw}\,, \tag{2.3.1a}$$

$$\alpha(\mathbf{Tv}) = (\alpha\mathbf{T})\mathbf{v} = \mathbf{T}(\alpha\mathbf{v})\,, \tag{2.3.1b}$$

$$(\mathbf{v}+\mathbf{w})\mathbf{T} = \mathbf{vT}+\mathbf{wT}\,, \tag{2.3.1c}$$

$$\alpha(\mathbf{vT}) = (\alpha\mathbf{v})\mathbf{T} = (\mathbf{vT})\alpha\,, \tag{2.3.1d}$$

where $\mathbf{v}$ and $\mathbf{w}$ are arbitrary vectors and α is an arbitrary real number. Notice that the tensor $\mathbf{T}$ can operate on its right [e.g., (2.3.1a), (2.3.1b)] or on its left [e.g., (2.3.1c), (2.3.1d)] and that, in general, operation on the right and the left is not commutative

$$\mathbf{Tv} \neq \mathbf{vT} \quad \text{Lack of commutativity}\,. \tag{2.3.2}$$

Zero Tensor of Order M:
The zero tensor of order M is denoted by $\mathbf{0}(M)$ and is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{0}(M-1)$ is the zero tensor of order $M-1$

$$\mathbf{0}(M)\mathbf{v} = \mathbf{v}\,\mathbf{0}(M) = \mathbf{0}(M-1)\,. \tag{2.3.3}$$

Notice that these tensors are defined inductively starting with the known properties of the real number 0 which is the zero tensor $\mathbf{0}(0)$ of order 0.

Addition and Subtraction:
The usual rules of addition and subtraction of two tensors $\mathbf{A}$ and $\mathbf{B}$ apply when the two tensors have the same order. It is emphasized that tensors of different orders cannot be added or subtracted.

To define the operations of tensor product, dot product and juxtaposition for general tensors it is convenient to first consider the definitions of these properties for the special case of the tensor product of a string of M ($M \geq 2$) vectors ($\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M$). Also, it is necessary to define the left transpose and right transpose of the tensor product of a string of vectors.