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# Preface

This book is a pedagogical introduction to supergravity. Supergravity is a gravitational field theory that includes supersymmetry (symmetry between bosons and fermions) and is a generalization of Einstein's general relativity. Supergravity provides a low energy effective theory of superstring theory, which has attracted much attention as a candidate for the unified theory of fundamental particles, and it is a useful tool for studying nonperturbative properties of superstring theory such as D-branes and string duality.

This work considers classical supergravities in four and higher dimensional spacetime with their applications to superstring theory in mind. More concretely, it discusses classical Lagrangians (or field equations) and symmetry properties of supergravities. Besides local symmetries, supergravities often have global non-compact symmetries, which play a crucial role in their applications to superstring theory. One of the main features of this book is its detailed discussions of these non-compact symmetries.

The aim of the book is twofold. One is to explain the basic ideas of supergravity to those who are not familiar with it. Toward that end, the discussions are made both pedagogical and concrete by stating equations explicitly. The other is to collect relevant formulae in one place so as to be useful for applications to string theory. They include the lists of possible types of spinors in each dimension, field contents of supergravities, and global symmetries of supergravities.

Most of the discussions are restricted to pure supergravities without matter couplings. An exception is a coupling to the super Yang–Mills multiplet in ten dimensions. Supergravities in lower than four dimensions are not considered. There are many other issues on supergravity which are not discussed in this book. For those issues, consult the references given at the end of Sect. 1.1.

The plan of the book is as follows. In Chap. 1, we first explain a role of supergravity in superstring theory briefly, and then review the formulations of the gravitational field and other fields coupled to it. In Chap. 2, we discuss supergravities in four dimensions in details. Much of the properties of supergravities in higher dimensions already appear in four dimensions. In Chap. 3, we discuss superalgebras and supermultiplets in general dimensions and give the lists of possible types of supergravities and their field contents. In Chap. 4, we consider global non-compact symmetries in supergravities, which are useful in understanding the structure of scalar fields. The non-compact symmetry in supergravities is sometimes realized as a duality symmetry of vector or antisymmetric tensor fields, which is a generalizations of the electric–magnetic duality in Maxwell's equations. In Chap. 5, we consider supergravities in higher dimensions. In particular, the Lagrangians (or field equations) and the symmetry properties of supergravities in 11 and 10 dimensions are discussed in details. In Chap. 6, we consider dimensional reductions of supergravities in eleven and ten dimensions to lower dimensions in order to understand the origins of the global non-compact symmetries. Finally, in Chap. 7, we consider gauged supergravities, which have minimal couplings to vector gauge fields, and massive supergravities similar to them. Notation and conventions used in this book are summarized in Appendix A. Formulae of gamma matrices and spinors in general dimensions are collected in Appendix B.

Saitama, April 2014 Yoshiaki Tanii

# Contents

# Chapter 1
# Introduction

## 1.1 Supergravity and Superstring

Supergravity is a theory of gravity which has supersymmetry, a symmetry between bosons and fermions. Supersymmetry in supergravity is a local symmetry like the gauge symmetry in the standard theory of particle physics. The gauge field of the local supersymmetry is the Rarita–Schwinger field, which represents a particle with spin $\frac{3}{2}$ called a gravitino. Supergravity also has a local symmetry under the general coordinate transformation, whose gauge field is the gravitational field.

An important role of supergravity is in its relation to superstring theory. Superstring theory is a candidate of the unified theory of fundamental particles including gravity. It is expected that superstring theory provides a consistent quantum theory of gravity since it has no ultraviolet divergence. Superstring theory contains massless states such as graviton and gauge particles in addition to infinitely many massive states. The mass scale of the theory is supposed to be very large, say the Planck energy scale, and only the massless states are important at low energy. Supergravity provides a low energy effective theory of the massless sector of superstring theory and can be used to study its low energy properties.

Superstring theory is an incomplete theory at present as its fundamental formulation is not known. As a consequence, it is difficult to study its non-perturbative properties, whose understanding is indispensable for applications to physics. In the 1990s there was considerable progress in non-perturbative understanding of string theory, which was possible by two important discoveries: D-branes [8] and string duality [4, 16].

D-branes are spatially extended objects appearing in string theory and correspond to solitons in field theories. As solitons are useful tools in non-perturbative study of field theories, D-branes can be used to study non-perturbative properties of string theory. At low energy, D-branes are represented by classical solutions of the effective supergravity theory. One can study various aspects of D-branes by using supergravity.

String duality is the relation among seemingly different string theories. Five supersymmetric string theories are known. They are type I, type IIA, type IIB superstring

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theories and two kinds of heterotic string theories, all of which are formulated in 10-dimensional spacetime. String duality provides relations among these superstring theories. Furthermore, it predicts another theory in 11-dimensional spacetime called M theory [11, 16] as a strong coupling limit of type IIA superstring theory. By string duality these six theories are intimately related, and it is conjectured that they are different aspects of one fundamental theory. String duality sometimes relates a theory at strong coupling and another theory at weak coupling. One can use this relation to study the former in terms of the latter, which may be studied by a perturbative method. Thus, string duality is extremely useful to understand non-perturbative properties of string theories. However, at present understanding of string theory, it is difficult to prove string duality directly in full string theory. On the other hand, the massless sectors of superstring theories are described by supergravities, for which complete field theoretic formulations are known at the classical level. One can obtain information on string duality by using supergravity. Indeed, supergravities containing scalar fields have a global symmetry of non-compact Lie group, whose discrete subgroup corresponds to the expected group of string duality.

In this book we consider classical supergravities in four and higher dimensional spacetime with their applications to superstring theory and M theory in mind. More concretely, we discuss classical Lagrangians (or field equations) of supergravities and their symmetry properties. In particular, we will discuss global non-compact symmetries in detail, which are related to string duality as mentioned above. We do not try to give complete references to the original work. For more complete references see [2, 10]. Other useful references on supergravity are [6, 13, 14]. For reviews of superstring theory and M theory, see [1, 3, 9, 17].

## 1.2 Gravitational Field

Supergravity contains various kinds of fields including the gravitational field. In the following we review the formulations of the gravitational field and other fields coupled to it. We consider field theories in general $D$-dimensional spacetime. Spacetime coordinates are denoted as $x^\mu$ ($\mu = 0, 1, \cdots, D-1$), where $x^0$ is time. It is convenient to use the variational principle to formulate field theories. The action $S$ is an integral of the Lagrangian (density) $\mathscr{L}$ over $D$-dimensional spacetime

$$S = \int d^D x\, \mathscr{L}, \qquad d^D x = dx^0 dx^1 \ldots dx^{D-1}. \tag{1.1}$$

The Lagrangian $\mathscr{L}$ is a function of the fields and their derivatives. The field equations are obtained by the least action principle for $S$.

There are two formulations of the gravitational field: the metric formulation and the vielbein formulation. To couple gravity to tensor fields such as scalar and vector fields, both formulations can be used. To couple gravity to spinor fields, the vielbein formulation should be used.

### 1.2.1 Metric Formulation

In the usual formulation of the gravitational theory a metric $g_{\mu\nu}(x)$ is used to describe gravity. Such a formulation is called the metric formulation. The Lagrangian for the gravitational field $g_{\mu\nu}(x)$ is

$$\mathscr{L} = \frac{1}{16\pi G}\sqrt{-g}\,(R - 2\Lambda)\,, \tag{1.2}$$

where $G$ is Newton's gravitational constant, $\Lambda$ is the cosmological constant and $g = \det g_{\mu\nu}$. The first and the second terms are called the Einstein term and the cosmological term, respectively. In the following we will set $16\pi G = 1$ for simplicity. The scalar curvature $R$ is defined from the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ as

$$\begin{aligned} &R = g^{\mu\nu} R_{\mu\nu}, \qquad R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu}, \\ &R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \end{aligned} \tag{1.3}$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$. The Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ is defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right). \tag{1.4}$$

This form of the Christoffel symbol is uniquely determined by the two conditions

$$\begin{aligned} &\text{metricity condition: } \partial_\lambda g_{\mu\nu} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\mu\rho} = 0, \\ &\text{torsionless condition: } \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}. \end{aligned} \tag{1.5}$$

By using the Christoffel symbol we can define the covariant derivative $D_\mu$ on tensor fields. For instance, the covariant derivatives on a contravariant vector field $V^\mu$ and a covariant vector field $V_\mu$ are defined as

$$\begin{aligned} D_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\rho} V^\rho, \\ D_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^{\rho}_{\mu\nu} V_\rho. \end{aligned} \tag{1.6}$$

By using the covariant derivative, the metricity condition in (1.5) can be written as $D_\lambda g_{\mu\nu} = 0$. The covariant derivatives of tensor fields transform covariantly under the general coordinate transformation given below.

The field equation of $g_{\mu\nu}(x)$ derived from the Lagrangian (1.2) by the variational principle is the Einstein equation with a cosmological term

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{1.7}$$

$T_{\mu\nu}$ is the energy–momentum tensor of matter fields. For now, $T_{\mu\nu} = 0$ as there is no matter field.

The action (1.1) for the Lagrangian (1.2) is invariant under the general coordinate transformation. The general coordinate transformation of the gravitational field $g_{\mu\nu}(x) \to g'_{\mu\nu}(x)$ corresponding to a change of the coordinates $x^\mu \to x'^\mu = x'^\mu(x)$ is determined by

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x). \tag{1.8}$$

For $x'^\mu = x^\mu - \xi^\mu(x)$, where $\xi^\mu(x)$ is an arbitrary infinitesimal vector function, the variation of the gravitational field $\delta_G g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x)$ is given by

$$\begin{aligned} \delta_G g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \\ &= D_\mu \xi_\nu + D_\nu \xi_\mu. \end{aligned} \tag{1.9}$$

The variation of the Lagrangian (1.2) under this transformation is a total divergence

$$\delta_G \mathscr{L} = \partial_\mu \left(\xi^\mu \mathscr{L}\right) \tag{1.10}$$

and therefore the action (1.1) is invariant when an appropriate boundary condition is imposed on the field at infinity.

Coupling to a Scalar Field

As a matter field, let us first consider a real scalar field $\phi(x)$. The Lagrangian of a free scalar field of mass $m$ coupled to the gravitational field is

$$\mathscr{L} = \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2\right]. \tag{1.11}$$

Under the general coordinate transformation (1.9) and

$$\delta_G \phi = \xi^\mu \partial_\mu \phi \tag{1.12}$$

the Lagrangian transforms to a total divergence as in (1.10) and the action is invariant.

### *1.2.2 Vielbein Formulation*

To couple gravity to spinor fields we have to use the vielbein formulation of gravity. In this formulation we introduce $D$ independent vectors $e_a{}^\mu(x)$ $(a = 0, 1, \ldots, D-1)$ at each point of spacetime, which are orthogonal to each other and have a unit length

$$e_a{}^\mu(x) e_b{}^\nu(x) g_{\mu\nu}(x) = \eta_{ab}, \qquad \eta_{ab} = \mathrm{diag}(-1, +1, \ldots, +1). \tag{1.13}$$

We also introduce the inverse matrix $e_\mu{}^a(x)$, which satisfies

$$e_\mu{}^a(x)e_a{}^\nu(x) = \delta_\mu^\nu, \qquad e_a{}^\mu(x)e_\mu{}^b(x) = \delta_a^b. \tag{1.14}$$

The field $e_\mu{}^a(x)$ is called the vielbein (vierbein or tetrad in four dimensions, fünfbein in five dimensions, etc.). From (1.13) and (1.14) we can express the metric in terms of the vielbein as

$$g_{\mu\nu}(x) = e_\mu{}^a(x)e_\nu{}^b(x)\eta_{ab}. \tag{1.15}$$

Therefore, we can use the vielbein $e_\mu{}^a(x)$ as dynamical variables representing the gravitational field. We choose $e_\mu{}^a$ such that the determinant $e = \det\, e_\mu{}^a$ is positive and therefore $\sqrt{-g} = e$.

For a given metric $g_{\mu\nu}$ the vielbein $e_\mu{}^a$ satisfying (1.15) is not uniquely determined. If $e_\mu{}^a$ satisfies (1.15), then

$$e'_\mu{}^a(x) = e_\mu{}^b(x)\Lambda_b{}^a(x), \qquad \Lambda_a{}^c(x)\Lambda_b{}^d(x)\eta_{cd} = \eta_{ab} \tag{1.16}$$

also satisfies (1.15) with the same $g_{\mu\nu}$. The vielbein has $D^2$ independent components while the metric has $\frac{1}{2}D(D+1)$ independent components. The difference $\frac{1}{2}D(D-1)$ is the number of independent components of $\Lambda_a{}^b$ in (1.16). The transformation (1.16) is called the local Lorentz transformation. A theory originally given in the metric formulation should be invariant under the local Lorentz transformation when it is rewritten in the vielbein formulation. Thus, the gravitational theory in the vielbein formulation has two local symmetries: the symmetries under the general coordinate transformation and the local Lorentz transformation.

We have now two kinds of vector indices: $\mu, \nu, \ldots$ and $a, b, \ldots$. To distinguish them, the indices $\mu, \nu, \ldots$ are called the world indices, while $a, b, \ldots$ are called the local Lorentz indices. These two kinds of indices are converted into each other by using the vielbein and its inverse, e.g., $V_a(x) = e_a{}^\mu(x)V_\mu(x)$, $V_\mu(x) = e_\mu{}^a(x)V_a(x)$. Raising and lowering of indices are done by using the metric $g^{\mu\nu}$, $g_{\mu\nu}$ for world indices and the flat metric $\eta^{ab}$, $\eta_{ab}$ for local Lorentz indices, e.g., as $V^\mu = g^{\mu\nu}V_\nu$, $V^a = \eta^{ab}V_b$. Tensor fields with local Lorentz indices transform under the local Lorentz transformation as in (1.16). They also transform under the general coordinate transformation as tensor fields according to the world indices they have. For instance, the general coordinate ($G$) and the local Lorentz ($L$) transformations of the vielbein $e_\mu{}^a(x)$ are

$$\delta_G e_\mu{}^a = \xi^\nu\partial_\nu e_\mu{}^a + \partial_\mu\xi^\nu e_\nu{}^a, \quad \delta_L e_\mu{}^a = -\lambda^a{}_b e_\mu{}^b, \tag{1.17}$$

where $\xi^\mu(x)$ and $\lambda_{ab}(x) = -\lambda_{ba}(x)$ are infinitesimal transformation parameters. The relation of $\lambda^a{}_b$ to $\Lambda^a{}_b$ in (1.16) is $\Lambda^a{}_b = \delta^a_b + \lambda^a{}_b$.

To construct the covariant derivative for the local Lorentz transformation we need a gauge field. The gauge field of the local Lorentz transformation is called the spin